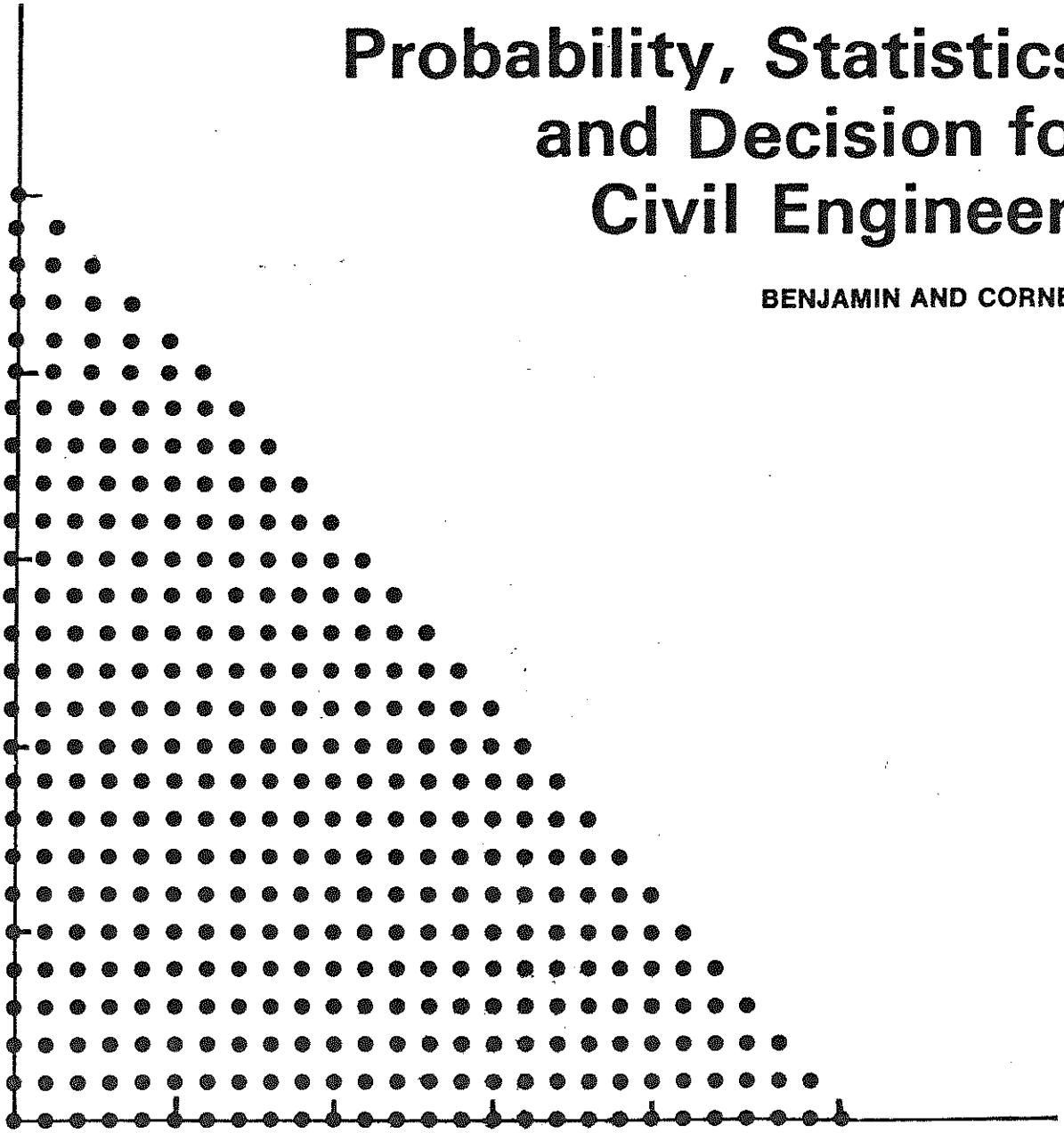
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SOLUTIONS MANUAL TO ACCOMPANY

Probability, Statistics, and Decision for Civil Engineers

BENJAMIN AND CORNELL



Probability, Statistics, and Decision for Civil Engineers

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Solutions Manual t/a PROBABILITY, STATISTICS AND DECISION FOR CIVIL ENGINEERS

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Important Errata in 1st Printing of

Probability, Statistics, and Decision for Civil Engineers

J. R. Benjamin and C. A. Cornell

McGraw-Hill Book Co., October, 1970, (Corrected in 2nd Printing)

These represent only the errata that may not be evident to the reader and hence might cause him confusion.

PAGE	LOCATION	NOW	CHANGE TO
23	P 1.10	Totals are incorrect.	
99	Eq. 2.2.65 and 66:	$F_D^n(C)$	$[F_D(C)]^n$ } Several other such difficult-to-interpret superscripts exist in the text.
105	Line 9	ac^{bx}	ae^{bx}
133	6 fm bottom	$Z = z$	$Z \leq z$
142	several places	6367	8567
163	Fig. Caption	(a)...$\rho < 0$	(a)...$\rho > 0$
167	line 6	2E[CD]-E[C]E[D]	2(E[CD]-E[C]E[D])
169	2 fm bottom	from Eq. 2.4.80	from Eq. 2.4.72
206	P 2.44	$2x/r_o$	$2x/r_o^2$
231	line 15	in y-1 trials	in w-1 trials
235	11 fm bottom	p^{-y}	p^y
249	line 14	$(\ )/(s-1)!$	$(\ )/(k-1)!$
"	line 16	$(s-1)!...\Gamma(s)$.	$(k-1)!!..\Gamma(k)$
259	11 from bottom	$m_x = 100 + 40$	$m_z = 100 + 40$
266	Eq. 3.3.30	$\exp(-1/2r^2\sigma^2)$	$\exp(1/2r^2\sigma^2)$
340	3 fm bottom	$= (0.7)(0.2)...= 0.24$	$= (0.7)(0.1)...= 0.17$
345	5 equations from middle of page	The effects of this numerical error propagate through pages 340-342. Interchange rows for columns of numerical values of coefficients of q_i^*'s. This changes values of q_i^*'s, e.g., $q_i^* = 0.52$.	
355	P 3.9	$t \to 0$	$\Delta t \to 0$
365	P 3.39 (twice on last three lines)	m_D	m_G
367	P3.46	line 4 interchange words "max" and "min"	
"	"	line 6 exactly	at least
"	"	line 7 change i's to j's and add	$\sum_{j=i}^{n}$
408	Several places	4.21	4.29
412	bottom	4.303	2.90
436	Fig. 4.3.4	s/n	$s/\sqrt{n}$
438	5	$\frac{s^2}{s_x^2} = \frac{n}{n-2}(\ )...$ subsequent numerical values change.	$\frac{s^2}{ns_x^2} = \frac{1}{n-2}(\ )...$
513	6	1, 1, 2, 2, 4	1, 1, 1.5, 2, 4
528	Fig. 5.1.2	interchange -100 and -400	
534	7	$E[u\|a_1] >$	$E[u\ a_1] <$
603	2	E[4-8x]	E[4-8X]
640	Eq. 6.3.28	$\binom{y}{s}$	$\binom{n}{s}$

CHAPTER 1

1.1 Deflection: $\bar{x}$ = 0.1417 in., s^2 = 0.000,209(in.)^2

Ultimate: $\bar{x}$ = 1847.5 lb., s^2 = 88,431 (lb.)^2

r = - 0.52

The average observed deflection exceeds by about 3s (see also Prob. 4.35) the computed deflection 0.098 in. The allowable load is nearly 5s below the mean ultimate. Deflection and load are lightly negatively correlated (see page 417). These statements are true of these 10 beams, not necessarily all possible beams.

1.2 Load at 1/16 in. yield: $\bar{x}$ = 3020 lb., s = 357 lb., v = 0.119
Load at 1/8 in. yield : $\bar{x}$ = 3693 lb., s = 297 lb., v = 0.081
Ultimate Load : $\bar{x}$ = 4085 lb., s = 338 lb., v = 0.083
r = 0.69 between load at 1/8 in. yield and ultimate.
Although considerable scatter is present, load at 1/8 in. yield and ultimate load appear to be correlated so that one can be used as a predictor of the other.

1.3 Straight cables: $\bar{x}$ = 1.7711, s^2 = 0.0559
Curved cables : $\bar{x}$ = 2.2542, s^2 = 0.0879

1.4 Note the wide variations in the appearance of the histograms as a function of plotting procedure.

1.6 Discharge: $\bar{x}$ = 13,353 acre-ft. s = 2131 acre-ft.
Recharge: $\bar{x}$ = 13,312 acre-ft. s = 2157 acre-ft.
r = 0.916

1.7 $\bar{x}$ = 70,248 cfs, $s^2 = 2.66 \times 10^9$ (cfs)^2
Relative frequency: 49/59 = 0.83

1.8 $\bar{x}$ = 12.64 in., s^2 = 0.126 $in.^2$, q_1 = - 0.134

1.9 $\bar{x}$ = 893 m^3/sec, s^2 = 187,616 $(m^3/sec)^2$

A histogram will be useful to the contractor since it indicates the type of variation in the area of concern. The decision is optimized by balancing cost of construction and expected loss from flooding.

1.10

	$\bar{x}$ sec.	s sec.	v
Wait and Exchange	0.718	0.665	0.93
Load	1.933	0.239	0.12
Haul	2.096	0.093	0.043
Dump and turn	0.563	0.0946	0.17
Return	2.53	0.1706	0.067
Total	8.13	0.598	0.074

Most variable step: Wait and Exchange

Sources of variability: Wait and Exchange, Load

Some combinations appear faster than others.

The standard deviation of the sum is smaller than the sum of the standard deviations of the parts. See Prob. 1.22. Note that the "total times" are independently recorded values, not totals of the individual times. The sample mean and standard deviation of these latter totals are 7.84 and 0.699.

1.11 $\bar{x}$ = 20.95 min.

s = 4.44 min.

g_1 = - 0.0263

g_2 = 2.82

1.12 $\bar{x}$ = 4.0

s^2 = 7.2

v = 0.67

One possibility for quality control is to make a running plot of the number of defectives against the mean of the data. See Prob. 4.6.

The cost of testing, cost of rejected pipe, and manufacturing cost must be considered.

1.13

	$\bar{x}$	s^2	s	v
Load	17.88	13.23	3.64	0.20
Swing	9.64	11.43	3.38	0.35
Dump	2.04	0.0384	0.196	0.096
Return	9.68	2.78	1.67	0.17
Total	39.24	27.22	5.22	0.13

The primary source of variability is swing. The most valuable summary statistic is the variance. The coefficient of variation is very sensitive to the magnitude of $\bar{x}$.

1.14 $\bar{x}$ = 861 s = 37.3

1.15 Southbound: $\bar{x}$ = 2.08 vehicles/30 sec. s^2 = 2.58 s = 1.59

Northbound: $\bar{x}$ = 3.08 vehicles/30 sec. s^2 = 4.12 s = 2.03

An automatic signal is best when the traffic flow is of constant moderate intensity. A large fraction of intervals with few cars will produce unnecessary delays.

1.16 $\bar{x} = 1773 \quad s^2 = 1{,}652{,}610$

1.17 $\bar{x} = 4577 \times 10^3$ gal per day $\quad s^2 = 365{,}731 \times 10^6$ (gal per day)^2
Demand exceeds capacity 69 per cent of the days recorded.

1.18 Diatomite: $\bar{x} = 6.786 \quad s^2 = 23.03$
Sand: $\bar{y} = 2.178 \quad s^2 = 1.986$

$r_{XY} = 0.132$

1.19
$$s_{XY} = \frac{1}{n}\sum(x_i-\bar{x})(y_i-\bar{y})$$
$$= \frac{1}{n}\sum(x_iy_i-\bar{x}y_i-x_i\bar{y}+\bar{x}\,\bar{y})$$
$$= \frac{1}{n}(\sum x_iy_i)- \frac{1}{n}\bar{x}n\bar{y} - \frac{1}{n}\bar{y}n\bar{x} + \frac{1}{n}\bar{x}\,\bar{y}n$$
$$= \frac{1}{n}(\sum x_iy_i)- \bar{x}\,\bar{y}$$

1.20 a. A scattergram of direction versus speed shows joint behavior. A study of direction for, say, 13 to 18 mph wind speeds involves simply a histogram of one quantity, direction. This histogram shows wind direction distribution for a given wind velocity. Note higher velocities are more likely to come from northwesterly directions.

b. Speed: $\bar{x} = 12.65$ mph, $s^2 = 11.56$ (mph)^2, $s = 3.40$ mph
Direction: $\bar{x} = 201.8°$, $s = 101.3°$
Note mean and variance can be computed from frequencies of data. Consider the sample mean:

$$\bar{x} = \frac{1}{n}\sum_{1}^{n} x_i$$

If there are r different possible values and there are m_j observations at value x_j, then

$$\sum_{i=1}^{n} x_i = \sum_{j}^{r} (m_1x_1+ m_2x_2 \ +\ldots m_jx_j\ldots+m_rx_r)$$

Let relative frequency $f_i = \frac{m_i}{n}$

$\therefore \quad \bar{x} = \sum(\frac{m_1}{n} x_1 + \frac{m_2}{n} x_2 + \ldots + \frac{m_r}{n} x_r)$

$$\bar{x} = \sum_{1}^{r} f_i x_i$$

and $$s^2 = \sum_{1}^{r} f_i x_i^2 - \bar{x}^2$$

This demonstrates that one will find identical values of $\bar{x}$ and s^2 no matter what the sample size if the relative frequencies are unchanged. In particular, s^2 does not decrease with increase in sample size.

c. The calm days should be omitted from the direction study. That is, the results are for wind direction <u>given</u> that it is blowing. The frequencies should be renormalized to sum to one by multiplying each by 100/98.

1.21 Recommend speed limit of 65 mph and 40 mph for passenger cars on Route 50 and Main St. respectively.

a. A dual speed limit for passenger cars and trucks on Route 50 appears reasonable.

b.

	$\bar{x}$, mph	s, mph
U.S. Route 50		
Local Passenger	56.4	8.1
Other Passenger	60.4	6.2
Trucks	48.4	7.6
All Vehicles	56.0	8.5
Main Street		
Northbound	34.7	4.7
Southbound	36.1	3.9
Total	35.2	4.4

c. Use the 15 per cent rule on Route 50. Political concerns will temper the Main St. speed limit. If traffic flows freely without hazards, the speed limit will tend to increase. If special hazards are present, pressure will exist toward a lower speed limit.

1.22 $t_i = x_i + y_i$ $\quad x_i$, variable 1; y_i, variable 2

$$s_T^2 = \frac{1}{n}\sum(t_i-\bar{t})^2$$

$$= \frac{1}{n}\sum[x_i+y_i - \frac{1}{n}\sum(x_i+y_i)]^2$$

$$= \frac{1}{n}\sum[x_i-\bar{x}+y_i-\bar{y}]^2$$

$$= \frac{1}{n}\sum[(x_i-\bar{x})^2 + 2(x_i-\bar{x})(y_i-\bar{y}) + (y_i-\bar{y})^2]$$

$$= s_1^2 + s_2^2 + 2s_{12} = s_1^2 + s_2^2 + 2r_{12}s_1s_2$$

For a given s_1 and s_2, the maximum value of s_T^2 occurs when r_{12} is a maximum or $r_{12} = 1$:

$$s_T^2 \leq s_1^2 + 2s_1s_2 + s_2^2 \quad \text{or} \quad (s_1 + s_2)^2$$

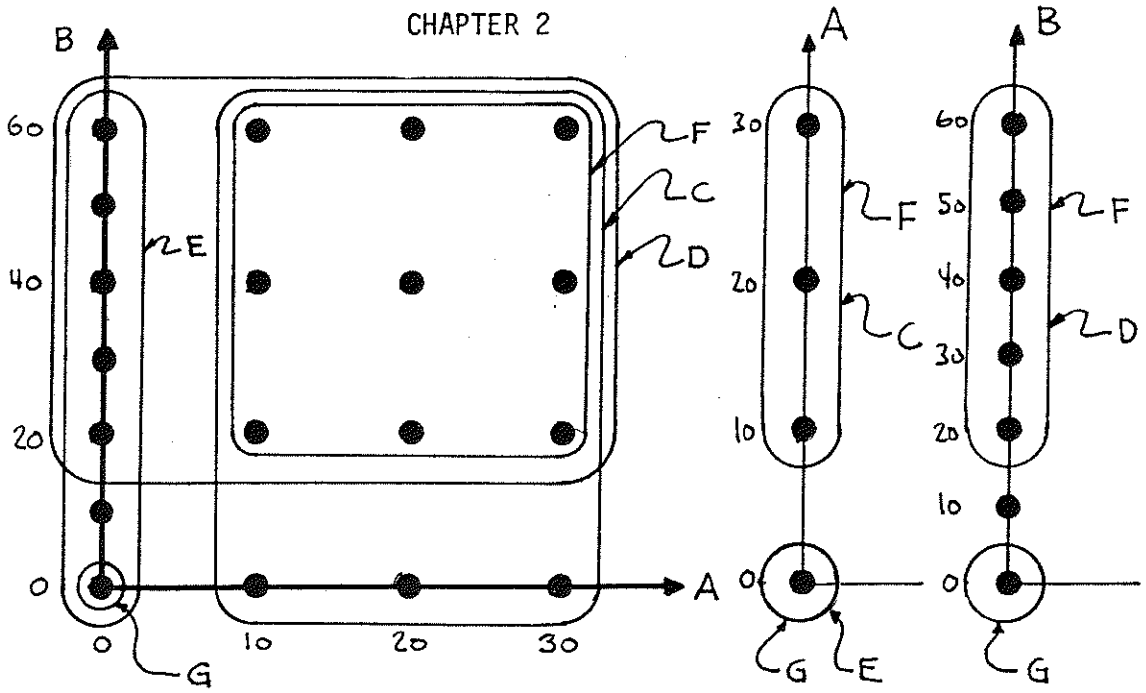
The minimum occurs for r_{12} minimum, $r_{12} = -1$:

$$s_T^2 \geq s_1^2 - 2s_1s_2 + s_2^2 \quad \text{or} \quad (s_1 - s_2)^2$$

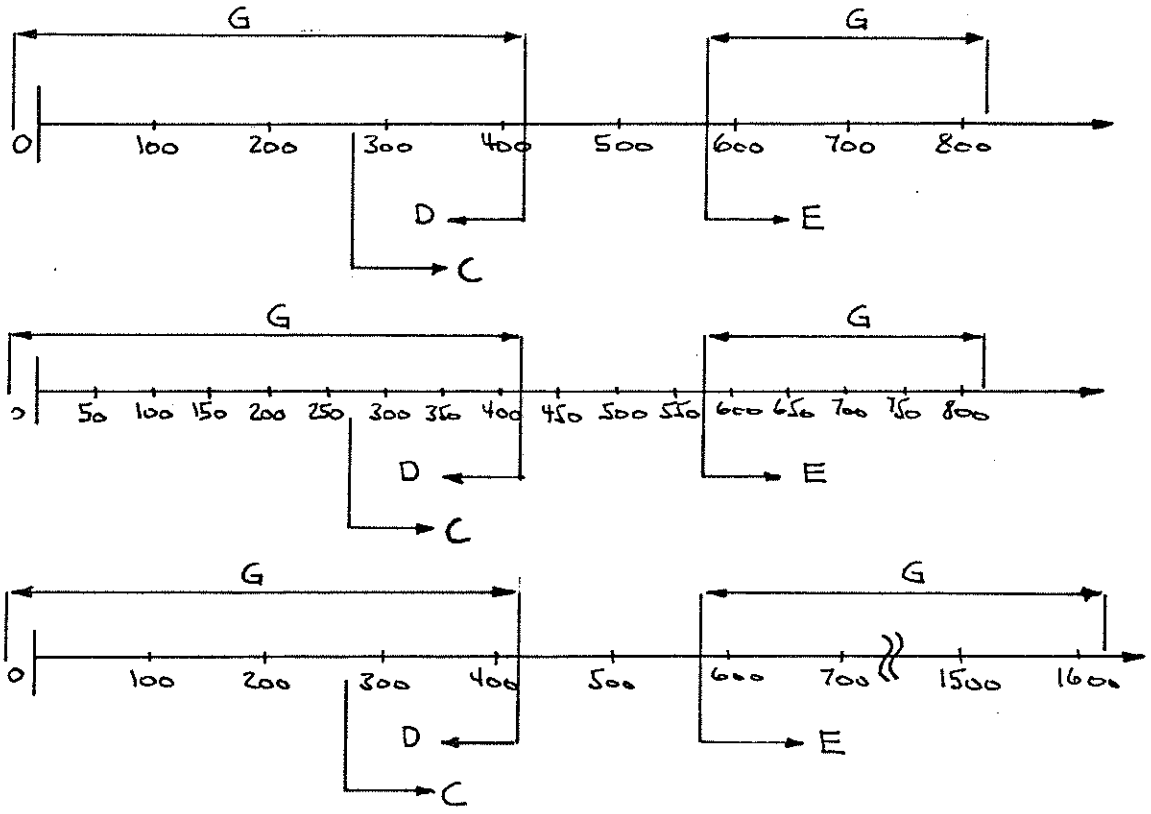
1.23 Note that negative blow counts are impossible but are included in the first interval so that it is centered on zero. The appropriate scale is relative frequency per unit blow count.

1.24 These data are naturally discrete whereas most engineering data, such as stream flows, are continuous.

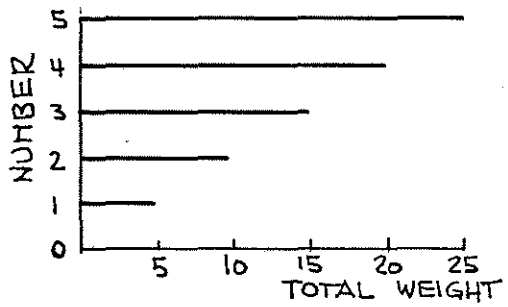
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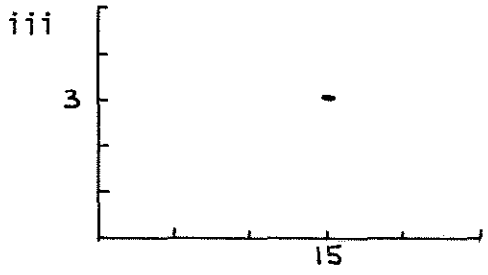

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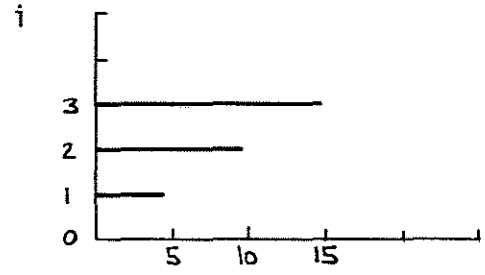
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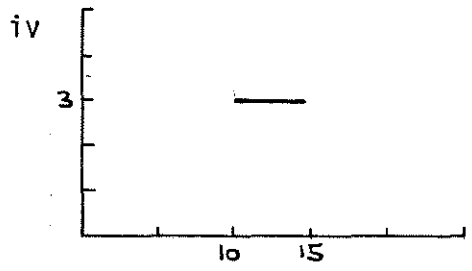

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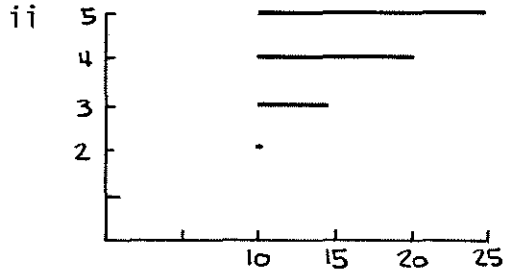
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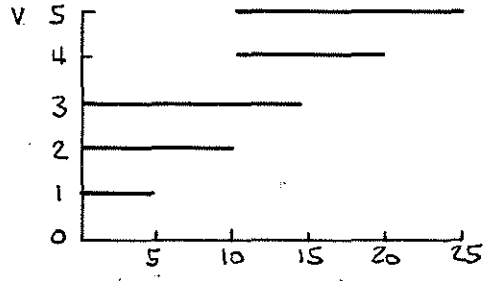

NUMBER
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10
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20
25
TOTAL WEIGHT

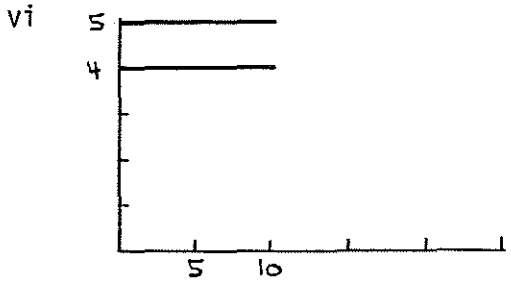

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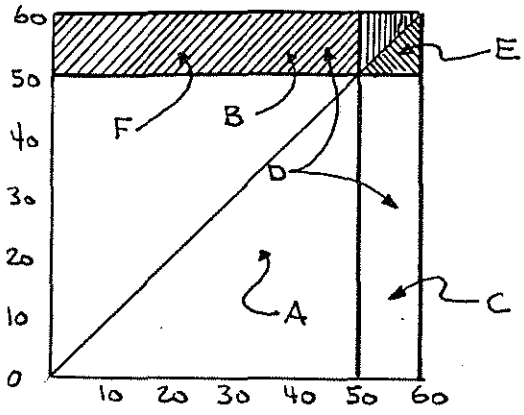

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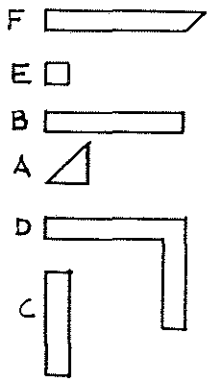

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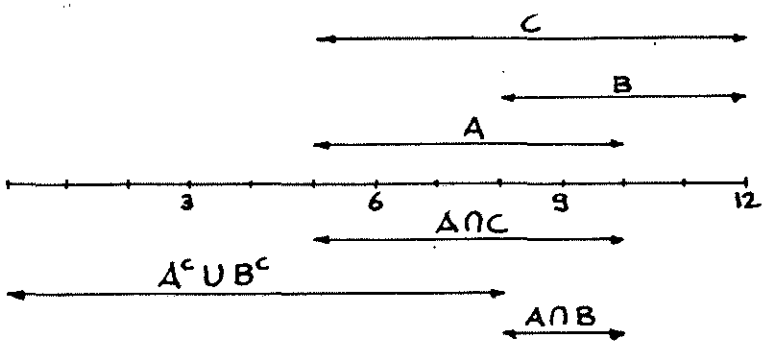

vii
ø
NULL EVENT

2.4


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D
A
C


F
E
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A
D
C

2.5 a)



b) $P[A \cap B] = P[A] + P[B] - P[AUB]$

$= 0.6 + 0.6 - 0.7 = 0.5$

$P[A^C] = 1 - P[A] = 0.4$

$P[BUA^C] = 1 - P[A \cap B^C] = 1 - (P[AUB] - P[B])$

$= 1 - (0.7 - 0.6) = 0.9$

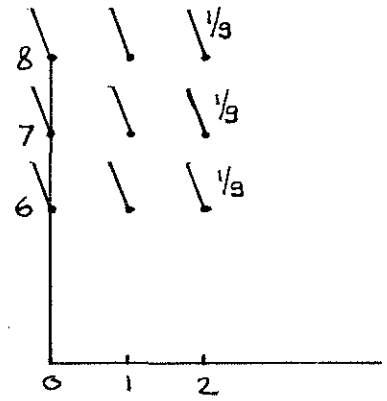
c) $P[A|B] = \frac{P[A \cap B]}{P[B]} = \frac{0.5}{0.6} = 5/6$

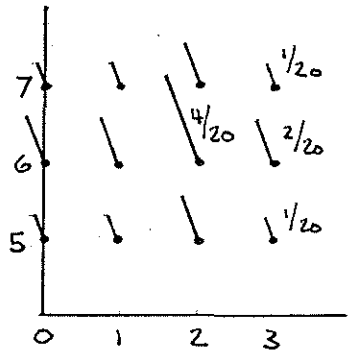
$P[B|A] = \frac{P[A \cap B]}{P[A]} = \frac{0.5}{0.6} = 5/6$

$P[B|A^C] = \frac{P[B \cap A^C]}{P[A^C]} = \frac{P[B] + P[A^C] - P[BUA^C]}{0.4}$

$= \frac{0.6 + 0.4 - 0.9}{0.4} = 1/4$

2.6 a)

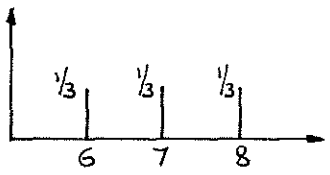




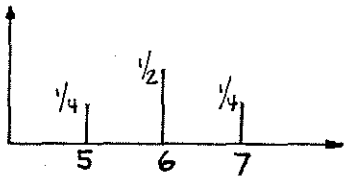
b) i) 3/9 ii) 2/9 iii) 2/9

c) i) 5/20 ii) 12/20 iii) 2/20 iv) 14/20 v) 12/20 vi) 12/20

d) A)

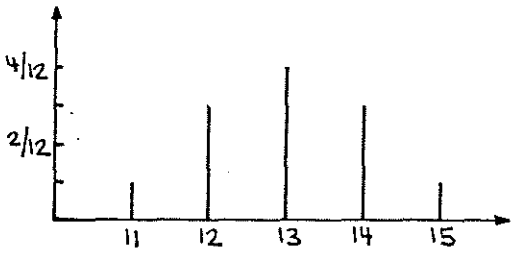


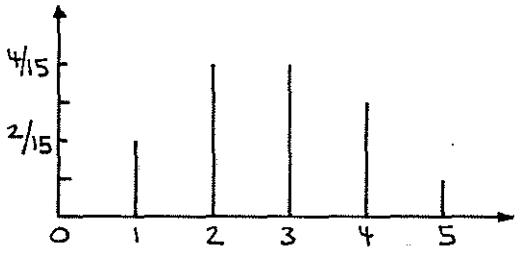
B)



e) No; because 1) the probability of both taking 7 hours is the product of the individual probabilities, and 2) an 8 hour and 6 hour combination will also take 14 hours.

f)





2.7 H =[high wind occurs]; E =[earthquake occurs]

a) $P[H \cap E] = P[H]P[E]$ by independence

$= 10^{-5} \times 10^{-8} = 10^{-13}$ Reasonable if this risk level is considered negligible or small compared to other undesigned for events (e.g., meteorite or airplane impact with building.)

b) $P[H \cup E] = P[H] + P[E] - P[H \cap E] = 10^{-5} + 10^{-8} - 10^{-13}$

$\cong 10^{-5} + 10^{-8} \cong 10^{-5}$

The assumption is reasonable if $P[A \cap B] << P[A]$ or $P[B]$, which it will be if, for example, A and B are independent and the sum of $P[A]$ and $P[B]$ is much less than unity. A common error is to apply this approximation when the sum is comparable to or even larger than unity!

c) Let H_i = [earthquake occurs during i^{th} minute of the year.]

P[no earthquake in year] = $P[H_1^c \cap H_2^c \ldots \cap H_{365x24x60}^c]$

$= (1 - P[H])^{365x24x60} = (1 - 10^{-8})^{5.25x10^5}$

Applying binomial expansion:

$\cong 1 - (10^{-8})(5.25x10^5) + \ldots$ or $\cong e^{-(10^{-8})(5.25x10^5)}$

$= 1 - 0.00525 = 0.99475$

P[no earthquake in 10 years] = $(1 - 10^{-8})^{5.25x10^6}$

$\cong 1 - 0.0525 + 1/2(0.0525)^2 = 1 - 0.052 + 0.001 = 0.949$

2.8 Total demand = 4 + number of high demands

Let N_i = [there are i high demands].

$P[N_0] = P[W_1W_1W_1W_1] = (0.3)^4 = 0.01$

$P[N_1] = 4P[W_2W_1W_1W_1] = 4(0.7)(0.3) = 0.08$

$P[N_2] = 6P[W_2W_2W_1W_1] = 6(0.3)^2(0.7)^2 = 0.26$

$P[N_3] = 4P[W_1W_2W_2W_2] = 4(0.3)(0.7)^3 = 0.41$

$P[N_4] = P[W_2W_2W_2W_2] = (0.7)^4 = 0.24$

Let x = number of units provided initially.

Final cost C is

$$C = \begin{cases} r\,x & \text{if } x \geq \text{no. of high demands} + 4 \\ r\,x + 2r(4 - x + \text{no. of high}) & \text{if } x < \text{no. of high demands} + 4 \end{cases}$$

in which r is the initial cost per unit.

The expected final cost is

$$E[C] = rx + 2r \sum_{i>x-4}^{4} P[N_i](4 - x + i)$$

For x = 8, E[C] = 8r

For x = 7, E[C] = 7r + 2r(1)(0.24) = 7.48r

For x = 6, E[C] = 6r + 2r[(1)(0.41) + (2)(0.24)] = 7.78r

The optimal value is 7 units initially for this cost ratio and probabilities.

2.9 $P[\text{reaching}] = P[\text{no hit in foot 1}]P[\text{no hit in foot 2}]\ldots P[\text{no hit in foot n}]$

$= (1 - p)^n$

$(1 - 1/20)^{10} = 0.5987$; $(1 - 1/20)^{20} = 0.3585$

$(1 - 1/10)^{10} = 0.3487$; $(1 - 1/10)^{20} = 0.1216$

2.10 a) $P[\text{inadequate supply}] = \sum P[\text{inadequate supply}|D_i]P[D_i]$

$= (0)(0.6) + (0.1)(0.3) + (0.5)(0.1) = 0.08$

b) $P[D_2|\text{inadequate supply}] = P[\text{inadequate supply}|D_2]P[D_2]/P[\text{inadequate supply}]$

$= (0.1)(0.3)/0.08 = 3/8$

c) $P[\text{pump failure}] = \sum P[\text{pump failure}|D_i]P[D_i]$

$= P[\text{pump failure}] \sum P[D_i] = P[\text{pump failure}] = 0.02$

d) $P[B_i|A \cap C_j] = P[B_i \cap A \cap C_j]/P[A \cap C_j]$

$= P[A|B_i \cap C_j]P[B_i \cap C_j]/P[A \cap C_j]$

$= P[A|B_i \cap C_j]P[B_i|C_j]P[C_j]/P[A|C_j]P[C_j]$

$= P[A|B_i \cap C_j]P[B_i|C_j]/P[A|C_j]$

E.g., $P[\text{inadequate supply}|\text{system failure} \cap D_2]$

$= \dfrac{P[\text{system failure}|\text{inadequate supply} \cap D_2]P[\text{inadequate supply}|D_2]}{P[\text{system failure}|D_2]}$

Assuming, as stated, that one and only one of the three possible modes of failure can occur, the 3 modes represent a mutually exclusive and exhaustive set of events <u>given a failure.</u> That is, the possibility of simultaneous pump failure and inadequate supply is ignored.

$P[\text{system failure}|D_2] = P[\text{system failure}|\text{inadequate supply} \cap D_2]$
$P[\text{inadequate supply}|D_2]$
$+ P[\text{system failure}|\text{pump failure} \cap D_2]$
$P[\text{pump failure}|D_2]$
$+ P[\text{system failure}|\text{overload} \cap D_2]P[\text{overload}|D_2]$

But $P[\text{overload}|D_2] = 0$

$P[\text{pump failure}|D_2] = 0.02$

$P[\text{inadequate supply}|D_2] = 0.1$

$P[\text{failure}|D_2] = (1)(0.1) + (1)(0.02) + 0 = 0.12$

Thus the probabilities of each cause are

$P[\text{inadequate supply}|\text{system failure}\cap D_2] = 0.1/0.12 = 5/6$

$P[\text{pump failure}|\text{system failure}\cap D_2] = 0.02/0.12 = 1/6$

$P[\text{overload}|\text{system failure}\cap D_2] = 0$

e) Given D_1

$P[\text{inadequate supply}|D_1] = P[\text{overload}|D_1] = 0$

$P[\text{pump failure}|D_1] = 0.02$

$P[\text{system failure}|D_1] = 0.02 + 0 + 0$

$P[\text{pump failure}|\text{system failure}\cap D_1] = 0.02/0.02 = 1$

$P[\text{inadequate supply}|\text{system failure}\cap D_1] = P[\text{overload}|\text{failure}\cap D_1] = 0$

2.11 a) $F_o = F_1 \cap F_2$

b) $P[F_o] = P[F_1]P[F_2]$; $P[F_o^c] = 1 - P[F_o] = 1 - (1 - P[F_1^c])(1 - P[F_2^c])$

$= P[F_1^c] + P[F_2^c] - P[F_1^c]P[F_2^c]$

Series: $F_o^c = F_1^c \cap F_2^c$; $P[F_o^c] = P[F_1^c]P[F_2^c]$

In former case $P[F_o^c]$ is larger than $P[F_1^c]$, while in latter case it is less. Former system is more reliable.

c) $P[F_o] = 10^{-2}$

d) $P[F_o] = P[F_1]P[F_2|F_1] = 10^{-1} \times 2 \times 10^{-1} = 2 \times 10^{-2}$

2.12 Using event notation (random variable notation is simpler),

let $Q_j^{(i)}$ = [a fraction j of the dam is amount of water in dam at start of i^{th} year]

$A_j^{(i)}$ = [a fraction j in dam before inflow (after release) during i^{th} year]

$R_j^{(i)}$ = [a fraction j released in i^{th} year]

$I_j^{(i)}$ = [a fraction j flows in in i^{th} year]

$P[A_{1/2}^{(1)}] = P[Q_1^{(1)}] = 1/3$

$P[A_{1/4}^{(1)}] = P[Q_{3/4}^{(1)}] = 1/3$

$P[A_0^{(1)}] = P[Q_{1/2}^{(1)} \cup Q_{1/4}^{(1)}] = 1/6 + 1/6 = 1/3$

The amount of water at the start of the 2nd year depends on $A_j^{(1)}$ and $I_j^{(1)}$

$P[Q_1^{(2)}] = P[A_{1/2}^{(1)} \cap I_{1/2}^{(1)}] = (1/3)(2/3) = 2/9$

$P[Q_{3/4}^{(2)}] = P[(A_{1/2}^{(1)} \cap I_{1/4}^{(1)}) \cup (A_{1/4}^{(1)} \cap I_{1/2}^{(1)})] = (1/3)(1/3) + (1/3)(2/3) = 1/3$

$P[Q_{1/2}^{(2)}] = P[(A_{1/4}^{(1)} \cap I_{1/4}^{(1)}) \cup (A_0^{(1)} \cap I_{1/2}^{(1)})] = (1/3)(1/3) + (1/3)(2/3) = 1/3$

$P[Q_{1/4}^{(2)}] = P[A_0^{(1)} \cap I_{1/4}^{(1)}] = (1/3)(1/3) = 1/9$

The values of $P[A_j^{(2)}]$ follow as above. Finally

$P[Q_1^{(3)}] = P[A_{1/2}^{(2)} \cap I_{1/2}^{(2)}] = (2/9)(2/3) = 4/27$

$P[Q_{3/4}^{(3)}] = (2/9)(1/3) + (1/3)(2/3) = 8/27$

$P[Q_{1/2}^{(3)}] = (1/3)(1/3) + (4/9)(2/3) = 11/27$

$P[Q_{1/4}^{(3)}] = (4/9)(1/3) = 4/27$

See Section 3.7 for a discussion of Markov chain models of which this is an example.

2.13

$P[Z_i] = \sum_j P[Z_i|\text{state } j]P[\text{state } j] = \{0.43, 0.44, 0.13\}$

$P[\text{state } j|Z_2] = P[Z_2|\text{state } j]P[\text{state } j]/P[Z_2] = \{0.455, 0.455, 0.090\}$

Another pit: use last set of state probabilities as prior probabilities of state, e.g.,

$P[50\% \text{ adequate}|Z_2 \text{ in third pit}] = P[Z_2|\text{state 1}]P[\text{state 1}]/P[Z_2]$

$= (0.4)(0.455)/P[Z_2]$

$P[\text{state } j|Z_2 \text{ in third pit}] = \{0.37, 0.54, 0.09\}$

The probability that the source is just adequate is improving, but slowly, with favorable outcomes. A single test pit carries very little information.

2.14 a) $\left(\frac{n_1}{n_1+m_1}\right)\left(\frac{n_3}{n_3+m_3}\right)^3 = \left(\frac{10}{80}\right)\left(\frac{10}{50}\right)^3 = \left(\frac{1}{8}\right)\left(\frac{1}{5}\right)^3 = 1/1000$

b) Note the question says "Given a rainy day now what is the probability that the next three will be dry?"

$$\left(\frac{m_3}{n_3+m_3}\right)\left(\frac{m_2}{n_2+m_2}\right)^2 = \frac{4}{5}\left(\frac{9}{10}\right)^2 = 0.648$$

2.15 a) $P[N>3] = 1 - \sum_{i=o}^{3} p_N(i)$

b) let N_i = [no. arriving during i^{th} cycle].

$P[\text{disturbed twice}|\text{no cars at end of previous green}]$

$= P[(N_1>3) \cap (N_1 + N_2>6)]$

$= P[N_1=4]P[N_2\geq 3] + P[N_1=5]P[N_2\geq 2]$

$+ P[N_1=6]P[N_2\geq 1] + P[N_1\geq 7]$

c) $P[N>3] = 0.143$; $P[(N_1>3) \cap (N_1 + N_2>6)] = 0.065$

The problem could also be solved given 1, 2, etc., cars at end of previous green phase. With a distribution of this number the marginal probability of being disturbed twice could be found. This problem can also be treated as a Markov chain.

2.16 a) All will be tested if any of first four is inadequate. Let A_i = [i^{th} is inadequate]. Several formal arguments will give the obvious answer of 2/3. For example,

$$P[\text{All are tested}] = P[A_1 \cup A_2 \cup A_3 \cup A_4] = P[A_1] + P[A_2 \cup A_3 \cup A_4] - P[A_1 \cap (A_2 \cup A_3 \cup A_4)]$$

Since at most one is inadequate

$P[A_1 \ (A_2 \cup A_3 \cup A_4)] = P[A_1]P[A_2 \cup A_3 \cup A_4 | A_1] = 0.$

Similarly

$P[A_2 \cup A_3 \cup A_4] = P[A_2] + P[A_3] + P[A_4]$

$P[A_1] = P[A_2] = P[A_3] = P[A_4] = 1/6.$ Therefore

$P[A_1 \cup A_2 \cup A_3 \cup A_4] = 2/3.$

Alternatively $P[\text{all tested}] = P[A_5^c \cap A_6^c] = P[A_5^c]P[A_6^c | A_5^c]$

$= (5/6)(4/5) = 2/3$

The second term is a result of knowing the bad cylinder is one of the five remaining cylinders.

b) Accepted if and only if two bad cylinders are 5^{th} and 6^{th}.

$P[A_5 \cap A_6] = P[A_5]P[A_6 | A_5] = (2/6)(1/5) = 1/15$

The second term is a result of knowing that one of five remaining cylinders is bad.

c) $P[\text{one or more bad}] = 1 - P[\text{none bad}]$

$= 1 - (P[\text{any one good}])^6 = 1 - (0.9)^6 = 0.47$

d) The 6 cylinders may contain 0 to 6 bad cylinders even if the batch is good. If 3 or more are bad the scheme will definitely test all cylinders and reject the batch. If there is either 0 or 1 bad cylinder, the batch will not be rejected no matter whether all are tested or not. If 2 are bad, the batch may or may not be rejected since they may not be found (part b). Therefore

$P[\text{accept}] = P[\text{0 or 1 bad cylinders}] + P[(\text{2 bad cylinders}) \cap (\text{not rejected})]$

= P[none bad] + P[1 bad] + P[2bad]P[not rejected|2 bad]

$= (0.9)^6 + 6(0.1)(0.9)^5 + 15(0.1)^2(0.9)^4(1/15)$

= .53 + .35 + .01 = 0.89

These terms require a generalization of Eq. 2.2.18 discussed in Section 3.1.2. For a "bad batch" only the probabilities of 0, 1, and 2 change.

$P[\text{accept}] = (0.7)^6 + 6(0.3)(0.7)^5 + (15)(0.3)^2(0.7)^4(1/15)$

$= 0.118 + .302 + 0.021 = 0.441$

In general one might have several (or a continuous spectrum of) possible values of p associated with different kinds or degrees of unsatisfactory batches. Given a probability distribution on these values and a solution like that above for the probability of accepting for each value, a marginal probability of accepting an unsatisfactory batch (with unspecified p) can be found.

2.17 This is a counting problem.

$P[10 \text{ seats}] = \sum_i P[10 \text{ seats}|i \text{ cars}]P[i \text{ cars}]$

P[10 seats|1 car] = 0;

P[10 seats|2 cars] = P[2 in first car]P[8 second] + P[8 first]P[2 second]

= (1/3)(1/3) + (1/3)(1/3) = 2/9

P[10 seats|3 cars] = P[4 first]P[4 second]P[2 third]
+ P[4 first]P[2 second]P[4 third]
+ P[2 first]P[4 second]P[4 third]

$= 3(1/3)^3 = 1/9$

$P[10 \text{ seats}|4 \text{ cars}] = 4(1/3)^4 = 4/81$

P[10 seats] = (2/9)(1/4) + (1/9)(1/4) + (4/81)(1/4) = 31/324 = 0.096

2.18 a) Independence of weighing processes implies

P[A|B] = P[A] and P[B|A] = P[B]

Note $C = (A \cap B) \cup (A^c \cap B^c)$

Thus $P[C] = P[A \cap B] + P[A^c \cap B^c] = P[A]P[B] + P[A^c]P[B^c]$

$= (1/2)(1/2) + (1/2)(1/2) = 1/2$

and only if
Given A, C occurs if/B occurs, i.e., $P[C|A] = P[B] = 1/2$
Therefore, for these particular numberical values, $P[C] = P[C|A]$
implying, by definition C and A are independent.
Similarly $P[C] = P[C|B]$, implying C and B are independent.
That is, A, B, and C are "pairwise" independent.

b) Mutual independence requires
$P[A \cap B \cap C] = P[A]P[B]P[C]$
But, here, $P[A \cap B \cap C] = P[A \cap B] = P[A]P[B]$ or 1/4, while
$P[A]P[B]P[C] = (1/2)^3 = 1/8$
Therefore, it is not necessarily true that pairwise independent events are also mutually independent.

c) Yes. If $P[A \cap B \cap C] = P[A]P[B]P[C]$
then $P[A \cap B] = P[A \cap B \cap C] + P[A \cap B \cap C^c]$
$= P[A]P[B]P[C] + P[A]P[B]P[C^c]$
$= P[A]P[B](P[C] + P[C^c]) = P[A]P[B]$

2.19 a) Events of interest are those of system performance, which in terms of reservoir level (H, high, and L, low) and number of down, N_i, tunnels are
$G = [\text{good performance}] = H \cap N_0$
$P = [\text{poor performance}] = (L \cap (N_1 \cup N_2 \cup N_3)) \cup (H \cap (N_2 \cup N_3))$
$M = [\text{marginal performance}] = (L \cap N_0) \cup (H \cap N_1)$

b) $P[N_1] = p_1(1 - p_2)(1 - p_3) + p_2(1 - p_1)(1 - p_3) + p_3(1 - p_1)(1 - p_2)$

c) $P[M] = q(1 - p_1)(1 - p_2)(1 - p_3) + (1 - q)P[N_1]$

d) $P[L|M] = P[L \cup M]/P[M] = q(1 - p_1)(1 - p_2)(1 - p_3)/P[M]$

2.20 Let $r = 1-p-q$

a) $p_Y(y) = \dfrac{n!}{y!(n-y)!} r^y (1-r)^{n-y}$ $\qquad y = 0, 1, 2, \ldots n$

If y turn west, n-y are left, and the conditional probability any one will turn east is P[east|not west] = P[east]/P[not west]

$= q/p+q$

Thus $p_{X|Y}(x,y) = \dfrac{(n-y)!}{x!(n-y-x)!} \left(\dfrac{q}{p+q}\right)^x \left(1 - \dfrac{q}{p+q}\right)^{n-y-x}$ $\qquad x = 0,1,\ldots n-y$, $0 \le y \le n$

b) $p_{X,Y}(x,y) = p_{X|Y}(x,y)p_Y(y)$

$= \dfrac{n!}{y!x!(n-y-x)!} r^y (1-r)^{n-y} \left(\dfrac{q}{p+q}\right)^x \left(\dfrac{p}{p+q}\right)^{n-y-x}$ $\qquad x = 0,1,\ldots n-y$, $0 \le y \le n$

2.21 Apply Bayes theorem

$$P[\text{shear failure}|\text{crack}] = \frac{P[\text{crack}|\text{shear}]P[\text{shear}]}{P[\text{crack}]}$$

in which, by the total probability theorem

P[crack] = P[crack|shear]P[shear] + P[crack|moment]P[moment]

= (0.8)(0.05) + (0.1)(0.95) = 0.135

P[shear failure|crack] = (0.8)(0.05)/0.135 = 8/27

P[moment failure|crack] = 19/27

Do not repair without further study since 19/27 > 8/27, even though a shear failure is "indicated" by the crack.

2.22

Let T_i = [true strength is i]; Z_i^j = [strength indicated by specimen j is i]

a) $P[Z_{1200}^1 \cap Z_{1200}^2 | T_{1200}] = P[Z_{1200}^1 | T_{1200}] P[Z_{1200}^2 | T_{1200}]$

$= (0.5)(0.5) = 0.25$

$P[Z_{1200}^1 \cap Z_{1000}^2 | T_{1200}] + P[Z_{1000}^1 \cap Z_{1200}^2 | T_{1200}] = 2(0.5)(0.3) = 0.30$

$P[Z_{1000}^1 \cap Z_{1000}^2 | T_{1200}] = (0.3)(0.3) = 0.09$

etc.

b) $P[T_{800} | Z_{1000}^1 \cap Z_{800}^2] = P[Z_{1000}^1 \cap Z_{800}^2 | T_{800}] P[T_{800}] / P[Z_{1000}^1 \cap Z_{800}^2]$

$= (0.2)(0.8)(\frac{2}{10}) / (0.2)(0.8)(\frac{2}{10}) + (0.5)(0.4)(\frac{5}{10}) + (0.3)(0.2)(\frac{3}{10})$

$= 0.21$

$P[T_{1000} | Z_{1000}^1 \cap Z_{800}^2] = 0.67; \quad P[T_{1200} | Z_{1000}^1 \cap Z_{800}^2] = 0.12$

Note that the problem does <u>not</u> state that the outcome was Z_{1000}^1 and Z_{800}^2; it might have been Z_{800}^1 and Z_{1000}^2.
It is easily shown that the same conclusion for the posterior probabilities is reached in either case, or, better still, in the case where the outcome is taken to be

$$(Z_{1000}^1 \cap Z_{800}^2) \cup (Z_{800}^1 \cap Z_{1000}^2).$$

c) i) After the first 1200 specimen, since $P[Z_{1200}^j | T_{800}] = 0$.

ii) We "start" now with "prior" probabilities $P[T_{800}] = 0.21$;

$P[T_{1000}] = 0.67$; and $P[T_{1200}] = 0.12$.

The conditional probabilities of n straight 1200 specimens are

$P[Z_{1200}^1 \cap Z_{1200}^2 \cap \ldots \cap Z_{1200}^n | T_{1200}] = P[Z_{1200}^j | T_{1200}]^n = (0.5)^n$

$P[Z_{1200}^j | T_{1000}]^n = (0.1)^n$

$P[Z_{1200}^j | T_{800}]^n = 0$

$$P[T_{1200} | Z_{1200}^1 \cap Z_{1200}^2 \cap \ldots \cap Z_{1200}^n] = \frac{(0.5)^n P[T_{1200}]}{(0.5)^n P[T_{1200}] + (0.1)^n P[T_{1000}]}$$

$$= \frac{(0.5)^n(0.12)}{(0.5)^n(0.12)+(0.1)^n(0.67)} = \frac{1}{1+(0.2)^n(5.6)}$$

Find lowest n such that this probability ≥ 0.90.

n =	1	0.48
	2	0.82
	3	0.96
	4	0.99

Conclusion: n = 3; note that the influence of the particular relative value of the prior probabilities (here 0.67/0.12 = 5.6) quickly becomes negligible.

2.23 The prior probability of a car leaving light 2 when light 1 is red is just the relative fraction of the time light 1 is red:

P["red"] = 1/(1 + 1.5 + 0.5) = 1/3; P["green"] = 1/2;

P["yellow"] = 1/6.

P["red"|delay] = P[delay|"red"]P["red"]/P[delay]

= (0.6)(1/3)/[(0.6)(1/3) + (0.2)(1/2) + (0.8)(1/6)] = 0.46

P["green"|delay] = 0.23; P["yellow"|delay] = 0.31

2.24 $P[\text{satisfactory}|\text{indicated defective}] = \frac{(0.05)(0.90)}{(0.05)(0.90)+(0.80)(0.10)} = 0.36$

Note, too, the probability of the other kind of machine mistake:

P[defective|indicated satisfactory] = 0.023

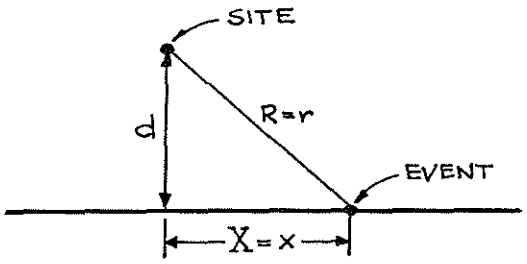
2.25 a)

x	$p_X(x)$	$y = 1000x+100\sqrt{x}$	$p_Y(y)$
1	0.05	1100	0.05
2	0.20	2141	0.20
3	0.20	3173	0.20
4	0.25	4200	0.25
5	0.20	5223	0.20
6	0.10	6245	0.10

b) $p_I(i) = \begin{cases} 0.05 & i = 10{,}000 - 1100 = 8900 \\ 0.05 & i = 10{,}000 - 2141 = 7859 \\ 0.05 & i = 12{,}000 - 2141 = 9859 \end{cases}$

etc.

2.26



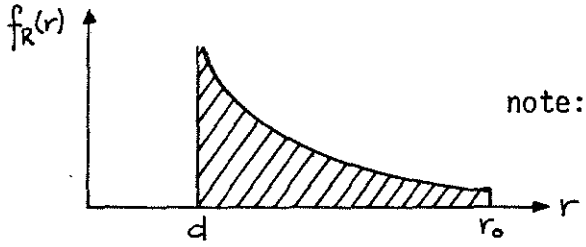
$|X|$ is uniformly distributed on the interval 0 to $\ell/2$

$$F_{|X|}(x) = P[|X| \le x] = \frac{x}{\ell/2} \qquad 0 \le x \le \ell/2$$

$$F_R(r) = P[R \le r] = P[\sqrt{|X|^2 + d^2} \le r] = P[|X|^2 + d^2 \le r^2]$$

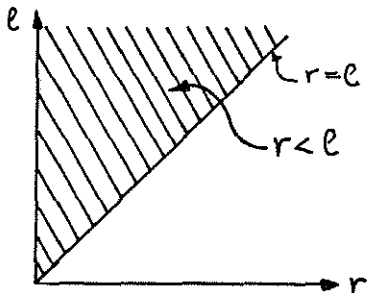
$$= P[|X| \le \sqrt{r^2 - d^2}\,] = F_{|X|}(\sqrt{r^2 - d^2}) = \frac{\sqrt{r^2-d^2}}{\ell/2} \qquad d \le r \le r_o$$

$$f_R(r) = \frac{dF(r)}{dr} = \frac{2r}{\ell}(r^2 - d^2)^{-1/2} \qquad d \le r \le r_o$$



note: $f_R(d) = \infty$

2.27



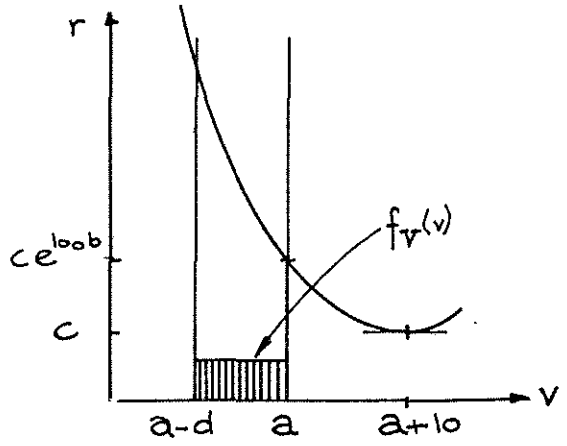
$$P_F = P[R < L] = \int_0^\infty dr \int_r^\infty d\ell\, f_{R,L}(r,\ell)$$

if $F_{R,L}(r,\ell) = 1/100 \quad 0 \le r \le 10; \quad 0 \le \ell \le 10$

$P_F = 1/2$

if $f_{R,L}(r,\ell) = 1/100$

2.28



In the range of r of interest ($a-d \le v \le a$) the relationship between R and V is monotonically decreasing. Therefore

$$f_R(r) = -\frac{dv}{dr} \qquad f_V(v) = -\frac{dv}{dr}\frac{1}{d}v \qquad a-d \le v \le a$$

$$v = g^{-1}(r) = a + 10 - \sqrt{\frac{1}{b}\ell n(r/c)}$$

note negative root is needed in range of interest.

$$dv/dr = -b^{-1}(1/2r)/\sqrt{\ell n(r/c)}$$

$$f_R(r) = \begin{cases} b^{-1}(1/2r)/\sqrt{\ell n(r/c)}\left(\frac{1}{d}\right)\left(a + 10 - \sqrt{\frac{1}{b}\ell n(r/c)}\right) \\ \qquad ce^{b100} \le r \le ce^{b(d+10)^2} \\ 0 \qquad \text{elsewhere} \end{cases}$$

2.29a) Knowing p_o:

$$p_N(n) = \frac{3!}{n!(3-n)!} p_o^n (1-p_0)^{3-n} \qquad n = 0,1,2,3$$

b) The result above can be interpreted as $p_{N|P}(n,p)$

Thus (using $p_{N,P}(n,p)$ to denote the joint distribution of a discrete variable N and a continuous variable P):

$$p_{N,P}(n,p) = p_{N|P}(n,p)\, f_P(p)$$

$$= \frac{3!}{n!(3-n)!} p^n (1-p)^{3-n} 3p^2 \qquad n = 0,1,2,3 \quad 0 \le p \le 1$$

c) The marginal distribution of N is

$$p_N(n) = \int_0^1 p_{N,P}(n,p)dp$$

$$= \frac{3!3}{n!(3-n)!} \int_0^1 p^{n+2}(1-p)^{3-n}\, dp$$

For n = 0

$$p_N(0) = \frac{3!3}{3!} \int_0^1 p^2(1-p)^3 dp = 3 \times 1/60 = 1/20$$

Similarly

$p_N(1) = 3/20$; $p_N(2) = 3/10$; $p_N(3) = 1/2$

d) $f_{P|N}(p,2) = p_{N,P}(2,p)/p_N(2) = 30p^4(1-p)$

$$P[P \leq p_o | N=2] = \int_0^{p_o} f_{P|N}(p,2)dp = p_o^5(6-5p_o)$$

e) One can improve the procedure's reliability (with respect to mistakenly "approving" bad batches) by increasing from 2 to 3 the number of specimens which must be O.K. before a batch is judged acceptable. The price paid is a higher number of good batches mistakenly rejected, i.e., lower reliability with respect to this second kind of mistake. Alternatively one can increase from 3 to say 6 the number of specimens tested in a sample, and then increase the fraction needed to pass from 2/3 to say 5/6. The price paid is a more expensive sampling procedure.

2.30 $F_X(x) = 1 - e^{-\lambda x}$

$G_X(x) = 1 - F_X(x) = e^{-\lambda x}$

$p = e^{-\lambda x_0}; \quad x_0 = -\frac{1}{\lambda} \ln p$

$$P[X \geq x_1 | X \geq x_0] = \frac{P[(X \geq x_1) \cap (X \geq x_0)]}{P[X \geq x_0]}$$

$$= \frac{P[X \geq x_1]}{P[X \geq x_0]} \qquad x_1 \geq x_0$$

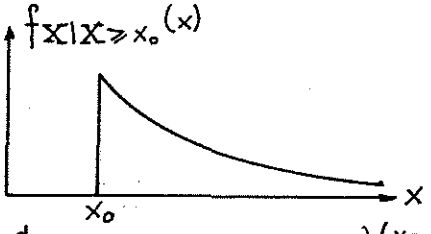
$$= e^{-\lambda x_1}/e^{-\lambda x_0} = e^{-\lambda(x_1 - x_0)}$$

$p = e^{-\lambda(x_1 - x_0)}$

$x_1 - x_0 = -\frac{1}{\lambda} \ln p$

$x_1 = x_0 - \frac{1}{\lambda} \ln p = -\frac{2}{\lambda} \ln p$

$F_{X|X \geq x_0}(x) = 1 - P[X \geq x_1 | X \geq x_0] = 1 - e^{-\lambda(x_1 - x_0)} \qquad x_1 \geq x_0$



$$f_{X|X \geq x_0}(x) = \frac{d}{dx} F_{X|X \geq x_0}(x) = \begin{cases} \lambda e^{-\lambda(x_1 - x_0)} & x_1 \geq x_0 \\ 0 & x_1 \leq x_0 \end{cases}$$

Note this conditional distribution is of the same form as that of X, but shifted x_0 units in the positive direction.

$$m_{X|X \geq x_0} = \int x f_{X|X \geq x_0}(x) dx = \int_{x_0}^{\infty} x \lambda e^{-\lambda(x_1 - x_0)} dx = x_0 + \frac{1}{\lambda}$$

$$\sigma^2_{X|X \geq x_0} = \int (x - m_{X|X \geq x_0})^2 f_{X|X \geq x_0}(x) dx = \frac{1}{\lambda^2}$$

2.31 X = strain; Y = stress

$Y = k\sqrt{X} = g(X)$

$f_X(x) = c[1 - \cos 2\pi\frac{x}{r}] \qquad 0 \le x \le r$

Since X and Y are restricted to positive values the transformation is monotonic and one-to-one.

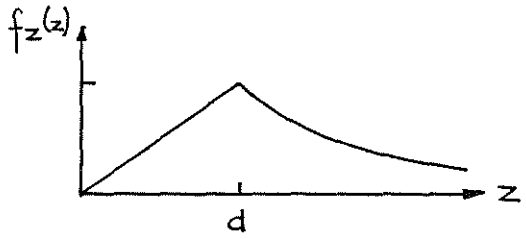
$X = g^{-1}(Y) = y^2/k^2$

$$\frac{dg^{-1}(y)}{dy} = 2y/k^2$$

$$f_Y(y) = \left|\frac{dg^{-1}(y)}{dy}\right| f_X(g^{-1}(y))$$

$$= \frac{2y}{k^2} c \left[1 - \cos\left(2\pi \frac{y^2}{rk^2}\right)\right] \quad 0 \le y \le k\sqrt{r}$$

2.32 a)



$$\int_0^\infty f_Z(z)dz = 1$$

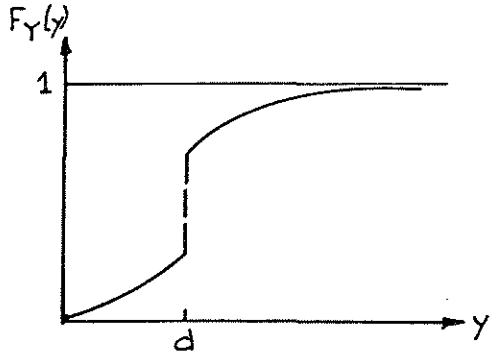
b) The relationship is not one-to-one.

For $y < d$, $F_Y(y) = P[Z \le y] = \frac{\lambda}{d(\lambda d+2)} y^2$

For $y = d$, $F_Y(y) = P[Z \le c+d] = 1 - P[Z > c+d]$

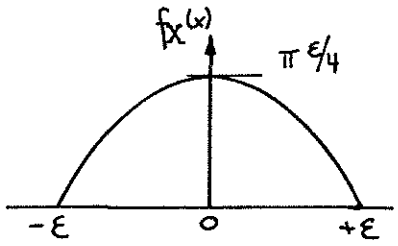
$$= 1 - \int_{c+d}^{\infty} \frac{2\lambda}{\lambda d+2} e^{-\lambda(z-d)}dz = 1 - \frac{2}{\lambda d+2} e^{-\lambda c}$$

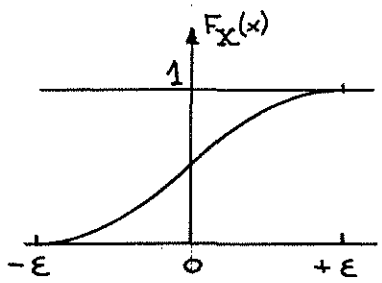
For $y > d$; $F_Y(y) = 1 - P[Z > c+y] = 1 - \frac{2}{\lambda d+2} e^{-\lambda(c+y-d)}$



2.33 a) $\int_{-\varepsilon}^{\varepsilon} N \cos \frac{\pi x}{2\varepsilon}\, dx = 1 = N\frac{4\varepsilon}{\pi}$; $N = \frac{\pi\varepsilon}{4}$

b)





c) $F_X(x) = \int_{-\varepsilon}^{x} \frac{\pi\varepsilon}{4} \cos \frac{\pi u}{2\varepsilon}\, du = 1/2(1 + \sin \frac{\pi x}{2\varepsilon})$

d) $P[X > \varepsilon/2] = 1 - F_X(\varepsilon/2) = 1 - 1/2(1 + \sin \pi/4)$
$= 0.147$

e) $P[|X| > \varepsilon/2] = P[X \leq -\varepsilon/2] + P[X \geq \varepsilon/2] = 0.294$

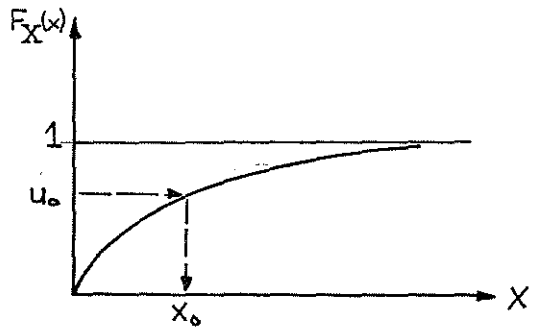
f) $f_X(x) = \frac{dF(x)}{dx} = \frac{d}{dx}[1/2(1 + \sin \frac{\pi x}{2\varepsilon})] = \frac{\pi}{4\varepsilon} \cos \frac{\pi x}{\varepsilon}$

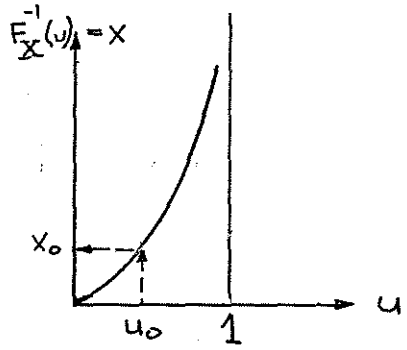
2.34 For smooth $F_X(x)$:

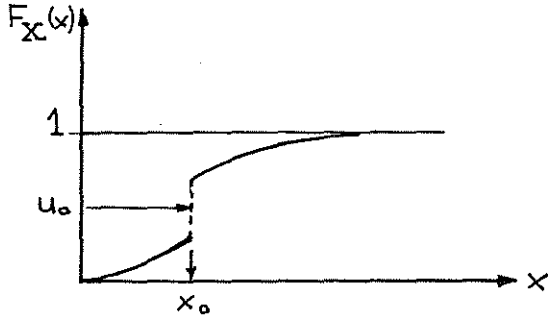
$$F_U(u) = P[U \leq u] = P[F_X(X) \leq u] = P[X \leq F_X^{-1}(u)]$$

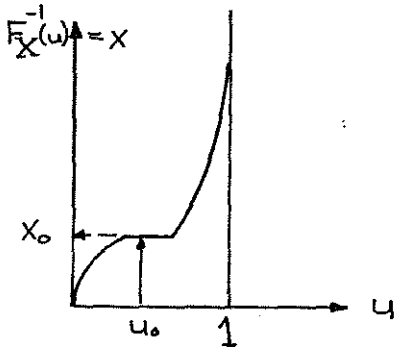
$$= F_X(F_X^{-1}(u)) = u \qquad 0 \leq u \leq 1$$

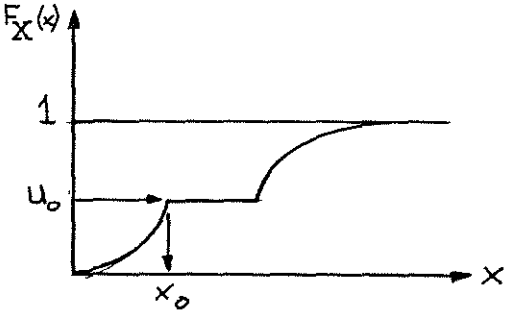
$$f_U(u) = dF_U(u)/du = 1$$

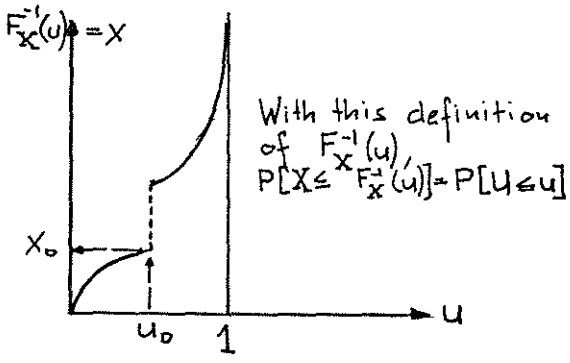












For $F_X(x) = 1 - e^{-x}$; $U = 1 - e^{-x}$; $X = F^{-1}(U) = -\ell n(1 - U)$

Therefore an observation of U equal to 0.51, implies an observation of X equal to $-\ell n(1 - 0.51) = 0.713$. Similarly for u = 0.20 and 0.98, x = 0.223 and 3.912.

2.35 a) $f_Y(y) = \left|\frac{1}{r}\right| f_X\left(\frac{y-y_o}{r}\right) = \frac{1}{r}\frac{c^{b+1}}{b!}\left(\frac{y-y_o}{r}\right)^b e^{-c(y-y_o)/r} \qquad y \geq y_o$

$$= \begin{cases} \frac{c^2}{r}\left(\frac{y-y_o}{r}\right) e^{-2(y-y_o)/r} & y \geq y_o \\ 0 & \text{elsewhere} \end{cases}$$

b) i) $E[Y] = E[y_o + rX] = y_o + rE[X]$

$$E[X] = \int_0^\infty x\frac{c^{b+1}}{b!} x^b e^{-cx} dx = \frac{c^{b+1}}{b!}\int_0^\infty x^{b+1} e^{-cx} dx$$

$$= \frac{c^{b+1}}{b!}\frac{(b+1)!}{c^{b+2}}\int_0^\infty \frac{c^{b+2}}{(b+1)!} x^{b+1} e^{-cx} dx$$

The integral eguals 1 since the integrand is a proper density function (like that of X with b+1 in place of b).

$$E[X] = \frac{b+1}{c} = \frac{1+1}{2} = 1$$

$$E[Y] = y_o + rE[X] = y_o + r$$

ii) $E[Y] = \int y f_Y(y) dy = \int_{y_o}^\infty y \frac{c^2}{r}\left(\frac{y-y_o}{r}\right) e^{-2(y-y_o)/r} dr$

Change variable $u = (y-y_o)/r$ and proceed as above.

$$E[Y] = y_o + r(b+1)/c$$

2.36 a) $E[X] = 1p + 0(1 - p) = p$

$Var[X] = E[X^2] - E^2[X] = (1^2 p) - p^2 = p(1 - p)$

$\frac{dVar[X]}{dp} = 0 = 1 - 2p$; maximum at $p = 1/2$

For p = 0.5; Var[X] = 0.25 For p = 0.1; Var[X] = 0.09

b) $E[X] = \sum x\, p_X(x) = \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^x}{x!}$

First term is zero, therefore

$$E[X] = \sum_{x=1}^{\infty} x \frac{e^{-\lambda}\lambda^x}{x!} = \sum_{x=1}^{\infty} \frac{e^{-\lambda}\lambda^{x-1}}{(x-1)!}\lambda = \lambda \sum_{x=1}^{\infty} \frac{e^{-\lambda}\lambda^{x-1}}{(x-1)!}$$

Let $u = x - 1$

$$E[X] = \lambda \sum_{u=0}^{\infty} \frac{e^{-\lambda}\lambda^u}{u!} = \lambda$$ since the sum is over a proper PMF

This is an example of evaluating such sums by manipulating them into recognizable sums over proper probability distributions; such a sum must equal unity.

$$E[X^2] = \sum_{x=0}^{\infty} x^2 \frac{e^{-\lambda}\lambda^x}{x!}$$

As above

$$E[X^2] = \lambda \sum_{x=1}^{\infty} x \frac{e^{-\lambda}\lambda^{x-1}}{(x-1)!}\lambda = \lambda \sum_{u=0}^{\infty} (u+1) \frac{e^{-\lambda}\lambda^u}{u!}$$

$$= \lambda\left(\sum_{u=0}^{\infty} u \frac{e^{-\lambda}\lambda^u}{u!} + \sum_{u=0}^{\infty} \frac{e^{-\lambda}\lambda^u}{u!} \right)$$

The first sum is the mean of a Poisson distribution, the second is unity.

$$E[X^2] = \lambda(\lambda+1) = \lambda^2 + \lambda$$

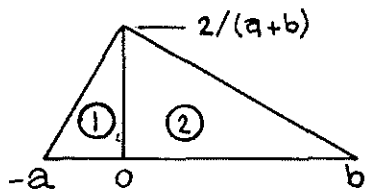
$$Var[X] = E[X^2] - E^2[X] = \lambda^2 + \lambda - \lambda^2 = \lambda$$

$$\sigma_X = \sqrt{\lambda}$$

For $\lambda = 0.5$; $m_X = 0.5$; $\sigma_X = 0.707$

For $\lambda = 2$; $m_X = 2$; $\sigma_X = 1.41$

c)



Area 1 = a/(a+b)

Area 2 = b/(a+b)

$E[X]$ is the first moment about zero:

$= \sum(\text{Area}_i)(\text{distance to centroid of area}_i)$ Centroids are at 1/3 of base from 0.

$= (\frac{a}{a+b})(-\frac{a}{3}) + (\frac{b}{a+b})\frac{b}{3} = 1/3(\frac{b^2-a^2}{b+a}) = 1/3(b-a)$

$E[X^2]$ is second moment about zero

By parallel axis theorem

$E[X^2] = \sum(\text{Area}_i)(\text{distance to centroid}_i)^2 + \sum(\text{Second-moment about own centroid})$

$= (\frac{a}{a+b})\frac{a^2}{9} + (\frac{b}{a+b})\frac{b^2}{9} + 2/9\frac{a^3}{a+b} + 2/9\frac{b^3}{a+b}$

$= \frac{a^3+b^3}{6(a+b)} = \frac{(a^2-ab+b^2)}{6(a+b)}$

$\sigma_X^2 = E[X^2] - E^2[X]$ (again the parallel axes theorem)

$= 1/18(a^2+ab+b^2)$

$\sigma_X = 1/3\sqrt{2}(a^2+ab+b^2)^{1/2}$

2.37 a) $P[\text{first occurrence in k}] = P[\text{no occ. in 1}|\text{occ. in 0}]P[\text{no occ. in 2}|\text{occ. in 0 and none in 1}]$

$\times P[\text{no occ. in 3}|\text{occ. in 0 and none in 1 or 2}]$

. . .

$\times P[\text{occ. in k}|\text{occ. in 0 and none in 1, 2, ...k-1}]$

$= (pa)(pa^2)(pa^3)\ldots(pa^{k-1})(1-pa^k)$

$= p^{k-1}a^{n_k}(1-pa^k)$

in which $n_k = 1 + 2 + 3 + \ldots + (k-1)$

b) $F_X(x) = 1-P[X > x] = 1-P[\text{no occurrences up to and including year x}]$

$= 1-(pa)(pa^2)(pa^3)\ldots(pa^x)$

$= 1-p^x a^{n_{x+1}}$

2.38 $\int_0^\infty a\, e^{-br} dr = \frac{a}{b} = 1 \quad$ or a=b

$f_R(r) = be^{-br}$

Note that $f_R(r)dr$ can be interpreted as the probability that an arbitrary person lives at radius r to r+dr, or as the (expected) fraction of all people living in that small radius interval (since this expected fraction is $\frac{1}{m}\sum_{i=1}^{m} P[i^{th}$ person lives in the interval] $= \frac{m}{m} f_R(r)dr$ where m is the total population). For large m the (random) fraction of the population living in the interval is with high probability very close to the expected fraction, by the Law of Large Numbers, Problem 2.43.

$\int_{r_o}^\infty f_R(r)dr = 1-0.75 = e^{-br_o}$; or $r_o = -\frac{1}{b}\ln(0.25) = 1.386/b$

a) $\int_0^\infty r f_R(r)dr = 1/b$ This mean radius would vary from city to city, of course.

b) The conditional distribution, given A, i.e., that a resident lies with 0 to 1.386/b from the center is,

$$f_{R|A}(r) = \begin{cases} \frac{f_R(r)}{P[A]} = \frac{f_R(r)}{0.75} & 0 \le r \le 1.386/b \\ 0 \text{ elsewhere} \end{cases}$$

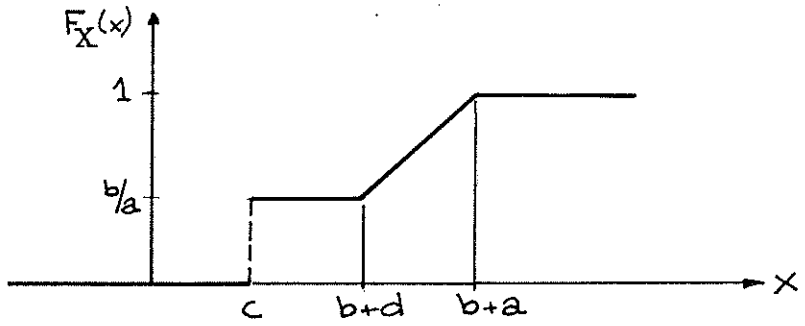
$$\int_0^\infty r f_{R|A}(r)dr = 4/3 \int_0^{1.386/b} rbe^{-br} dr = 4/3b(1-(1+1.386)e^{-1.386})$$

$$= 0.55/b$$

2.39 $x = g(t) = \begin{cases} c & 0 \le t \le b \\ d+t & t \ge b \end{cases}$

$F_T(t) = \int_0^t f_T(u)du = \frac{t}{a} \quad 0 \le t \le a$

Working with the CDF of X, consider different values of x:



$$F_X(x) = P[X \le x] = \begin{cases} 0 & x < c \\ P[T \le b] = \frac{b}{a} & c \le x \le b+d \\ P[T \le x-d] = \frac{x-d}{a} & b+d \le x \le a+d \\ 1 & x \ge a+d \end{cases}$$

X is mixed random variable. If T takes on any value between 0 and b, X will equal c. If T takes on a value greater than b, X = T + d.

2.40 a) 0.11

b) 0.49

c), d)

$p_X(x) =$	0.01	x=1.5	$0 = p_{X\|Y}(x, 1\ 3/4\pi)$
	0.16	x=5.5	1/11
	0.34	x=10	3/11
	0.36	x=15.5	4/11
	0.11	x=21.5	2/11
	0.02	x=28	1/11

e) $p_{Y|X\geq 21.5}(y) =$

1/13	$y=1/4\pi$
1/13	$y=3/8\pi$
1/13	$y=9/8\pi$
1/13	$y=5/4\pi$
1/13	$y=1\ 1/8\pi$
1/13	$y=1\ 1/2\pi$
3/13	$y=1\ 3/8\pi$
3/13	$y=1\ 3/4\pi$
1/13	$y=1\ 5/8\pi$

2.41 $P[T \geq 30] = \sum_{i=1,2,3} P[T \geq 30|\text{Type i is first}]P[\text{Type i is first}]$

$P[T \geq 30|\text{Type i is first}] = P[X_i \geq 10] = 1 - F_{X_i}(10)$

in which X_i is a random variable with distribution

$f_{X_i}(x) = \lambda_i e^{-\lambda_i x}$, or $1 - F_{X_i}(x) = e^{-\lambda_i x}$.

Thus

$$P[T \geq 30] = e^{-10(1/3)}(1/2) + e^{-10(1/5)}(1/3) + e^{-10(1/8)}(1/6)$$

$$= (.036)(1/2) + (.135)(1/3) + (0.288)(1/6)$$

$$= 0.018 + 0.045 + 0.048 = 0.111$$

The control plan could be improved in reliability by basing the decision on the type of truck as well as the time observed. A "slow", type C truck is quite likely (0.288) to take more than 30 minutes even if the operation is running as planned.

2.42 $\sigma^2 = \int_{-\infty}^{\infty}(x-m)^2 f(x)dx$

$$= \int_{-\infty}^{m-h\sigma}(x-m)^2 f(x)dx + \int_{m-h\sigma}^{m+h\sigma}(x-m)^2 f(x)dx + \int_{m+h\sigma}^{\infty}(x-m)^2 f(x)dx$$

Since $(x-m)^2 f_X(x)$ is non-negative, we can drop this term and say

$$\sigma^2 \geq \int_{-\infty}^{m-h\sigma}(x-m)^2 f(x)dx + \int_{m+h\sigma}^{\infty}(x-m)^2 f(x)dx$$

For $x > m+h\sigma$, $x-m \geq h\sigma$ and $(x-m)^2 \geq (h\sigma)^2$. Similarly for $x < m-h\sigma$, $(x-m)^2 \geq (h\sigma)^2$; therefore replacing $(x-m)^2$ by $h^2\sigma^2$ can only increase the right hand side:

$$\sigma^2 \geq \int_{-\infty}^{m-h\sigma} h^2\sigma^2 f(x)dx + \int_{m+h\sigma}^{\infty} h^2\sigma^2 f(x)dx$$

The right hand side is simply $h^2\sigma^2(P[X \leq m-h\sigma] + P[X \geq m+h\sigma])$

Therefore, dividing by σ^2, and combining the probability statements

$$P[|X-m| \geq h\sigma] \leq 1/h^2$$

$m = 5000$ and $\sigma = 1000$

$1/h^2 = 0.5$; $h = \sqrt{2}$; $5000 \pm 1000\sqrt{2}$
$1/h^2 = 1-0.75$; $h = 2$; 5000 ± 2000
$1/h^2 = 1-0.9$; $h = \sqrt{10}$; $5000 \pm 2000\sqrt{10}$
$1/h^2 = 1-0.99$; $h = 10$; $5000 \pm 10{,}000$

2.43 $E[Y] = E[\frac{1}{n}\sum X_i] = \frac{1}{n}E[\sum X_i] = \frac{1}{n}\sum E[X_i] = \frac{1}{n}\sum m = \frac{nm}{n} = m$

$Var[Y] = Var[\frac{1}{n}\sum X_i] = \frac{1}{n^2} Var[\sum X_i] = \frac{1}{n^2}\sum Var[X_i] = \frac{n\sigma^2}{n^2} = \sigma^2/n$

$\sigma_y = \sigma/\sqrt{n}$

Chebyshev's inequality (Eq.(2.4.11)) says

$$P[|Y-m| \leq \varepsilon] \geq 1 - \frac{\sigma_Y^2}{\varepsilon^2}$$

$$P[|Y-m| \leq \varepsilon] \geq 1 - \frac{\sigma^2}{n\varepsilon^2}$$

For arbitrary choice of (small) ε, the probability that Y lies within ε of m can be made arbitrarily close to unity by increasing n. So for large n Y is very likely to lie "close" to m. In particular, if the X_i's are (0,1) Bernoulli variables, then Y is the relative frequency of successes in n events and it is close to E[X] or p, the probability of success with high probability if the number of trials is large. In this sense probabilities and relative frequencies of events are related.

2.44 X and Y independent implies:

$$f_{X,Y}(x,y) = f_X(x)\, f_Y(y) = \frac{3}{32r_o}(9 - y)^2 X \qquad \begin{matrix} 0 \le x \le r_o \\ 5 \le y \le 9 \end{matrix}$$

$$P[((x \le 1/2r_o) \cap (y > 8))] =$$

$$\int_8^9 \int_o^{1/2r_o} f_{X,Y}(x,y)dxdy = \int_8^9 \int_o^{1/2r_o} \frac{3}{32r_o}(9 - y)^2 X \, dxdy$$

$$= r_o^2/256$$

2.45 m = mean strength

s = specified strength

σ = standard deviation

k = number of standard deviations by which s must lie below m

$$m = s + k\sigma$$

$$m = s + kVm$$

$$m = \frac{s}{1-kV}$$

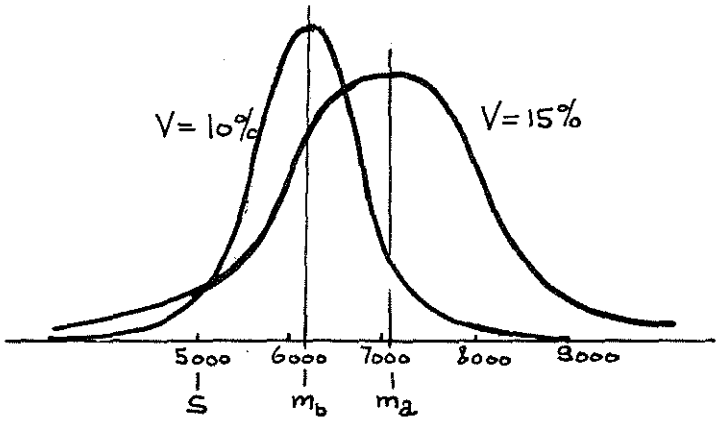
Note the impossibility of achieving a specification of conservatism k if $V \ge 1/k$.

s = 5000 psi; $V_a = .15$; $V_b = .10$; $k = 2$

$$m_a = \frac{5000}{1-0.3} = 7143 \text{ psi}$$

$$m_b = \frac{5000}{1-0.2} = 6250 \text{ psi}$$

$$m_a - m_b = 893 \text{ psi}$$



2.46 $M = A\,F_Y\left(D - K\frac{A\,F_Y}{F_c\,B}\right)$

$m_A = 1.5\ in^2$	$V_A = 0.05$	$\sigma_A = 0.075\ in^2$
$m_{F_Y} = 42\ ksi$	$V_{F_Y} = 0.07$	$\sigma_{F_Y} = 2.94\ ksi$
$m_D = 20\ in$	$V_D = 0.015$	$\sigma_D = 0.3\ in$
$m_B = 10\ in$	$V_B = 0.02$	$\sigma_B = 0.2\ in$
$m_{F_c} = 5\ ksi$	$V_{F_c} = 0.15$	$\sigma_{F_c} = 0.75\ ksi$
$m_K = 0.6$	$V_K = 0.1$	$\sigma_K = 0.06$

$$E(M) = E[g(X_i)] \simeq g[E(X_i)]$$

$$m_M \simeq (1.5)(42)\left[20 - (0.6)\frac{(1.5)(42)}{(5)(10)}\right]$$

$$= 63[20 - .756]$$

$$m_M = 1210\ \text{in.kips.}$$

$$Var[M] \simeq \sum_{i=1}^{n}\left\{Var[X_i]\left[\frac{\partial\, g(x_i)}{\partial\, x_i}\bigg|_m\right]^2\right\}$$

For example

$$\frac{\partial\, g}{\partial\, A}\bigg|_m = F_Y\left(D - K\frac{2AF_Y}{F_c B}\right)\bigg|_m = 42\left(20 - \frac{2(1.5)(42)}{0.6(5)(10)}\right) = 776$$

$$\frac{\partial\, g}{\partial F_Y}\bigg|_m = A\left(D - K\frac{2AF_Y}{F_c B}\right)\bigg|_m = 27.7$$

2.46 (continued)

$\left(\frac{\partial g}{\partial A}\right)^2 = 603000$ $\quad$ $\text{Var}[A] = 0.00563$

$\left(\frac{\partial g}{\partial F_Y}\right)^2 = 768$ $\quad$ $\text{Var}[F_Y] = 8.65$

$\left(\frac{\partial g}{\partial K}\right)^2 = 6320$ $\quad$ $\text{Var}[K] = 0.0036$

$\left(\frac{\partial g}{\partial F_c}\right)^2 = 253$ $\quad$ $\text{Var}[F_c] = 0.563$

$\left(\frac{\partial g}{\partial D}\right)^2 = 3970$ $\quad$ $\text{Var}[D] = 0.09$

$\left(\frac{\partial g}{\partial B}\right)^2 = 63.2$ $\quad$ $\text{Var}[B] = 0.04$

$$\text{Var}[M] = (0.00563)(603000) + (8.65)(768) + (0.0036)(6320) + (0.563)(253) + (0.09)(3970) + (0.04)(63.2)$$

$$= 3390 + 6650 + 22.8 + 142.5 + 357.3 + 2.5$$

$$\text{Var}[M] = 10{,}565(\text{in.kips})^2$$

$$V_M = \frac{\sqrt{10{,}565}}{1210} = 0.085$$

Even though V_{F_c} is the largest coefficient of variation, A, F_Y, and d contribute most significantly to Var[M]. Therefore engineers wishing to reduce Var[M] should concentrate on lowering the variance of steel yield force and steel placement, not concrete strength. It is easily shown that depth and concrete strength are dominant for shallow beams or slabs (assuming σ_D does not change). Compare V_M with those of the components. It is approximately equal to $\sqrt{V_A^2 + V_{F_Y}^2 + V_D^2}$ which would be a good approximation if $M \cong (\text{constant})AF_YD$, which is approximately true for such beams.

2.47 $Y = (X - m_X)/\sigma_X$

$$Y = \frac{X}{\sigma_X} - \frac{m_X}{\sigma_X}$$

a) $$E[Y] = E\left[\frac{X}{\sigma_X}\right] - E\left[\frac{m_X}{\sigma_X}\right] = \frac{1}{\sigma_X} E[X] - \frac{m_X}{\sigma_X}$$

but $E[X] = m_X$

$$E[Y] = \frac{1}{\sigma_X} m_X - \frac{m_X}{\sigma_X} = 0. \quad \text{Q.E.D.}$$

$$Var[Y] = Var\left[\frac{X}{\sigma_X}\right] - Var\left[\frac{m_X}{\sigma_X}\right] = Var\left[\frac{X}{\sigma_X}\right]$$

we know $Var[aX] = a^2 Var[X]$ where a is a constant

$$Var[Y] = Var\left[\frac{X}{\sigma_X}\right] = \frac{1}{\sigma_X^2} Var[X]$$

and $Var[X] = \sigma_X^2$ by definition

$$Var[Y] = \frac{1}{\sigma_X^2}(\sigma_X^2) = 1. \quad \text{Q.E.D.}$$

b) $$Cov\left[\frac{X-m_X}{\sigma_X}, \frac{Z-m_Z}{\sigma_Z}\right] = E\left[\left(\frac{X-m_X}{\sigma_X}\right)\left(\frac{Z-m_Z}{\sigma_Z}\right)\right] - E\left[\frac{X-m_X}{\sigma_X}\right] E\left[\frac{Z-m_Z}{\sigma_Z}\right]$$

$$= E\left[\left(\frac{X-m_X}{\sigma_X}\right)\left(\frac{Z-m_Z}{\sigma_Z}\right)\right]$$

$$Cov\left[\frac{X-m_X}{\sigma_X}, \frac{Z-m_Z}{\sigma_Z}\right] = E\left[\frac{XZ}{\sigma_X\sigma_Z} - \frac{m_X Z}{\sigma_X\sigma_Z} - \frac{m_Z X}{\sigma_X\sigma_Z} + \frac{m_X m_Z}{\sigma_X\sigma_Z}\right]$$

$$= \frac{1}{\sigma_X\sigma_Z} E[XZ] - \frac{m_X}{\sigma_X\sigma_Z} E[Z] - \frac{m_Z}{\sigma_X\sigma_Z} E[X] + \frac{m_X m_Z}{\sigma_X\sigma_Z}$$

2.47 b) continued

$$= \frac{Cov[X,Z]}{\sigma_X \sigma_Z} = \rho_{X,Z} \qquad \text{Q.E.D.}$$

2.48 $m[X] = m_R m_Y = (0.659)(7480) = 4920$ psi

$$Var[X] = \sigma_R^2\sigma_Y^2 + m_R^2\sigma_Y^2 + m_Y^2\sigma_R^2$$

$$= (0.165)^2(0.15)^2(7480)^2 + (0.659)^2(0.15)^2(7480)^2 + (7480)^2(0.165)^2$$

$$= 3.44 \times 10^4 + 56.0 \times 10^4 + 152 \times 10^4$$

$$= 2.11 \times 10^6$$

$$\sigma_X = 1460 \text{ psi}$$

2.49 a) $Z = aX + bY$

$W = cX + dY$

$Cov[Z,W] = E[Z,W] - E[Z]E[W]$

$ZW = (aX + bY)(cX + dY) = acX^2 + bcXY + adXY + bdY^2$

$Cov[Z,W] = ac\ E[X^2] + (bc + ad)\ E[XY] + bd\ E[Y^2] - E[aX + bY]E[cX + dY]$

$$= ac(\sigma_X^2+m_X^2) + (bc+ad)(\sigma_{XY}+m_Xm_Y) + bd(\sigma_Y^2+m_Y^2)$$

$$- (am_X+bm_Y)(cm_X+dm_Y)$$

$$Cov[Z,W] = ac\sigma_X^2 + bd\sigma_Y^2 + ad\sigma_{XY} + bc\sigma_{XY}$$

b) Z and W uncorrelated if $Cov[Z,W] = 0$

$$ac\sigma_X^2 + bd\sigma_Y^2 + ad\sigma_{XY} + bc\sigma_{XY} = 0$$

This is one requirement, so only one coefficient is not arbitrary.

Let a, b, c be arbitrary. Solve for d.

$$d = -c\left(\frac{b\sigma_{XY}+a\sigma_X^2}{a\sigma_{XY}+b\sigma_Y^2}\right)$$

e.g., if a=1, b=1, c=-1, then $d = \dfrac{\sigma_X^2+\sigma_{XY}}{\sigma_Y^2+\sigma_{XY}} = \dfrac{\rho+\sigma_X/\sigma_Y}{\rho+\sigma_Y/\sigma_X}$

2.49 (continued)

c) $X = a'Z + b'W$ $\qquad Z = aX + bY$

$Y = c'Z + d'W$ $\qquad W = cX + dY$

$$\frac{1}{b}Z = \frac{a}{b}X + Y$$

$$\frac{1}{d}W = \frac{c}{d}X + Y$$

$$\left(\frac{a}{b} - \frac{c}{d}\right)X = \frac{1}{b}Z - \frac{1}{d}W$$

$$X = \frac{d}{ad-bc}Z - \frac{b}{ad-bc}W$$

Similarly

$$Y = \frac{c}{bc-ad}Z - \frac{a}{bc-ad}W$$

Thus

$$a' = \frac{d}{ad-bc}$$

$$b' = \frac{-b}{ad-bc}$$

$$c' = \frac{-c}{ad-bc}$$

$$d' = \frac{a}{ad-bc}$$

Recall from b

a, b, c can be arbitrary $\qquad d = -c\left(\frac{b\sigma_{XY}+a\sigma_X^3}{a\sigma_{XY}+b\sigma_Y^2}\right)$

e.g., a=1, b=1, c=-1, $d = \frac{\rho+\sigma_X/\sigma_Y}{\rho+\sigma_Y/\sigma_X}$

$$a' = \frac{d}{d+1}$$

$$b' = \frac{-1}{d+1}$$

$$c' = \frac{1}{d+1}$$

$$d' = \frac{1}{d+1}$$

d) For the values of a, b, and c above, d = 1.
Z has mean a+b = 2 and standard deviation $0.25\sqrt{2}$
W has mean -1+d = 0 and standard deviation $0.25\sqrt{2}$
For each sample of X and Y generate a sample of a normal variable with mean 2 and standard deviation $0.2\sqrt{2}$ and one with mean 0 and the same standard deviation.

A sample of a normal variable with mean m and standard deviation σ can be obtained by taking $m+\sigma u$ where u is a sample of a standardized ($m=0$, $\sigma=1$) normal variable. This can be found directly from available tables or subroutines. Alternatively a standardized normal sample can be found (approximately, section 2.3.3 and 3.3.1) by adding 12, say, samples of a simple random number table or generator (uniform on 0 to 1). Since this sum has mean 12(0.5) = 6 and variance 12(1/12) = 1, the sample sum of 12 random numbers should be modified by subtracting 6 (and dividing by 1).

With a sample, z_i and w_i of random variables Z and W, a sample of an X and Y pair is found by

$$x_i = \frac{1}{2} z_i - \frac{1}{2} w_i$$

$$y_i = \frac{1}{2} z_i + \frac{1}{2} w_i$$

2.50 $E[C] = E[a(X+Y)^2] = E[a(X^2+2XY+Y^2)] = aE[X^2] + 2aE[XY] + aE[Y^2]$

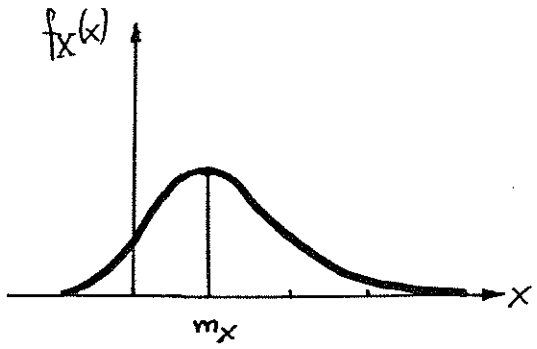
$= a(m^2+\sigma_X^2) + 2a(Cov[X,Y] + m_X m_Y) + a(m_Y^2+\sigma_Y^2)$

$= a(m_X^2+2m_X m_Y+m_Y^2) + a(\sigma_X^2+2\rho\sigma_X\sigma_Y+\sigma_Y^2)$

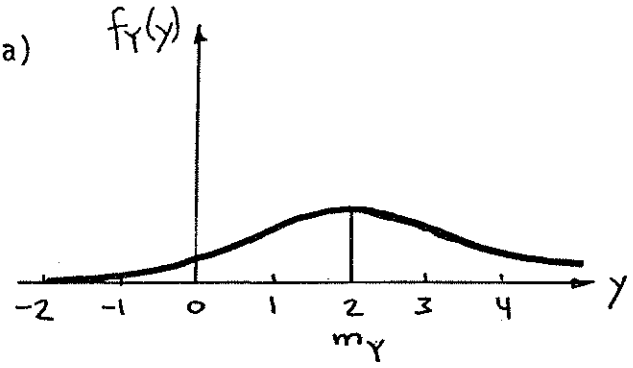
$= a(m_X+m_Y)^2 + a(\sigma_X^2+2\rho\sigma_X\sigma_Y+\sigma_Y^2)$

Note how this result compares with the "deterministic value", $a(m_X+m_Y)^2$. Note the special cases of $\rho = 0$ and $\rho = 1$.

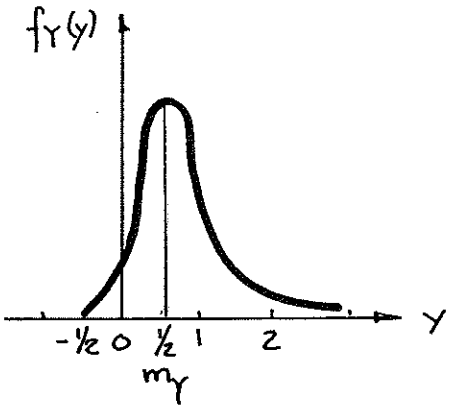
2.51


$f_X(x)$
X
m_X

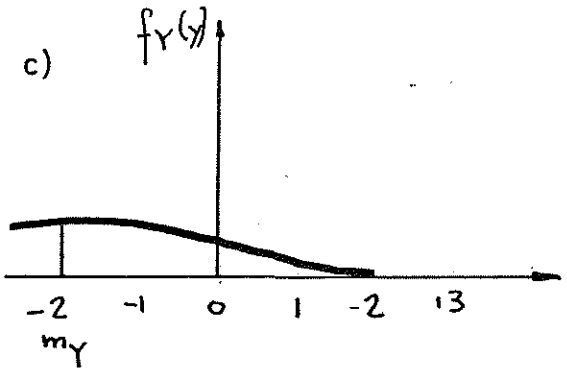
a)


$f_Y(y)$
-2 -1 0 1 2 3 4
y
m_Y

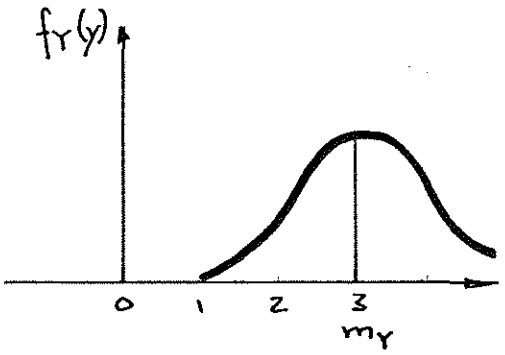
b)


$f_Y(y)$
-½ 0 ½ 1 2
Y
m_Y

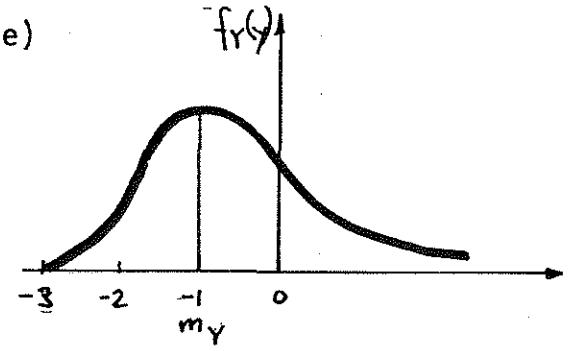
c)


$f_Y(y)$
-2 -1 0 1 -2 13
m_Y

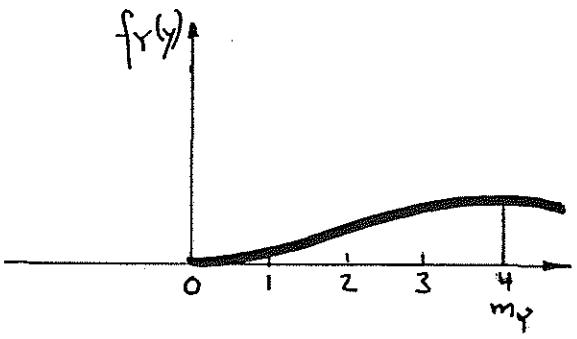
d)


$f_Y(y)$
0 1 2 3
m_Y

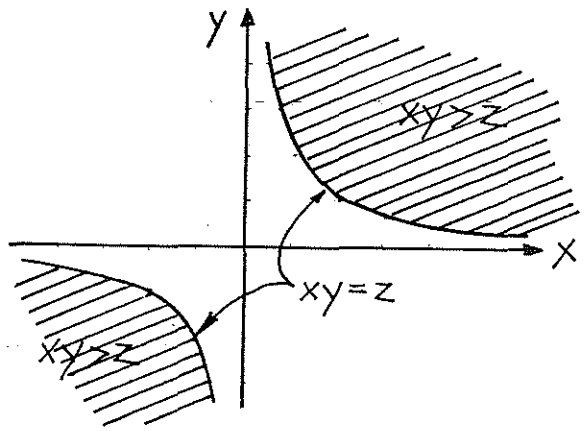
e)


$f_Y(y)$
-3 -2 -1 0
m_Y

f)


$f_Y(y)$
0 1 2 3 4
m_Y

2.52



$$F_Z(z) = P[Z \le z] = 1 - P[Z > z]$$

$$= 1 - \iint_{R_{shaded}} f_{X,Y}(x,y)\, dxdy$$

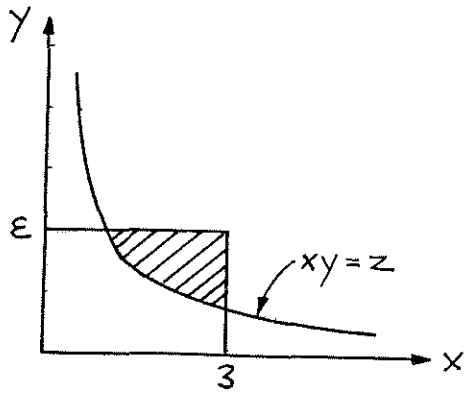
$$1 - \int_{-\infty}^{0} dy \int_{-\infty}^{z/y} dx\, f_{X,Y}(x,y) - \int_{0}^{\infty} dy \int_{z/y}^{\infty} dx\, f_{X,Y}(x,y)$$

$$f_Z(z) = \frac{dF_Z(z)}{dz} = -\int_{-\infty}^{0} f_Y(y)dy \frac{d}{dz} \int_{-\infty}^{z/y} f_X(x)dx - \int_{0}^{\infty} f_Y(y)dy \frac{d}{dz} \int_{z/y}^{\infty} f_X(x)\, dx$$

$$= -\int_{-\infty}^{0} f_Y(y)\, f_X\left(\frac{z}{y}\right) \frac{1}{y}\, dy + \int_{0}^{\infty} f_Y(y)\, f_X\left(\frac{z}{y}\right) \frac{1}{y}\, dy$$

$$= \int_{-\infty}^{\infty} f_Y(y)\, f_X\left(\frac{z}{y}\right) \left|\frac{1}{y}\right|\, dy$$

$$f_Z(z) = \int_{0}^{\varepsilon} \frac{1}{\varepsilon} f_X\left(\frac{z}{y}\right) \frac{1}{y}\, dy$$



Substitute for $f_X(x)$ which equals $\frac{2}{9}x$ for $0 \le x \le 3$.

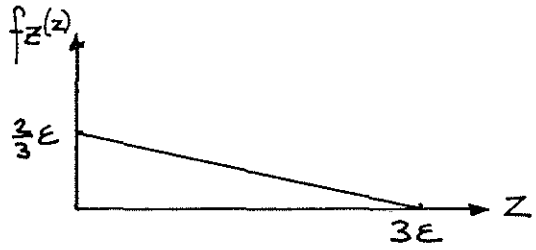
For $x=z/y$, this implies $0 \le z/y \le 3$, or $z/3 \le y \le \infty$.

2.52 (continued)

$$f_Z(z) = \int_{z/3}^{\varepsilon} \frac{1}{\varepsilon} \frac{2}{9} \left(\frac{z}{y}\right) \frac{1}{y}\, dy$$

$$= \frac{2z}{9\varepsilon} \int_{z/3}^{\varepsilon} \frac{dy}{y^2}$$

$$= \frac{2}{3\varepsilon} \left(1 - \frac{z}{3\varepsilon}\right) \quad 0 \le z \le 3\varepsilon$$



b) $\mathrm{Var}[XY] = E[X^2Y^2] - (E[XY])^2$

Since X and Y are independent, X^2 and Y^2 are independent

$$\mathrm{Var}[XY] = E[X^2]E[Y^2] - (E[X]E[Y])^2$$

$$= (m_X^2+\sigma_X^2)(m_Y^2+\sigma_Y^2) - m_X^2 m_Y^2$$

$$= \sigma_X^2\sigma_Y^2 + m_X^2\sigma_Y^2 + m_Y^2\sigma_X^2$$

c) $\mathrm{Var}[Z] = E[Z^2] - E^2[Z]$

$$E[Z] = \int z f_Z(z)\, dz = \frac{2}{3\varepsilon} \int_0^{3\varepsilon} \left(1 - \frac{z}{3\varepsilon}\right) z\, dz = \varepsilon$$

$$E[Z^2] = \int z^2 f_Z(z)\, dz = 3\varepsilon^2/2$$

$$\mathrm{Var}[Z] = 3\varepsilon^2/2 - \varepsilon^2 = \frac{1}{2}\varepsilon^2$$

Second method requires calculation of m_X, m_Y, σ_X^2, and σ_Y^2 and substitution into the results in b.

2.54 $$F_Y(y) = \int_{R_Y} f_X(x)\, dx$$

where R_Y is region where $ce^x \le y$

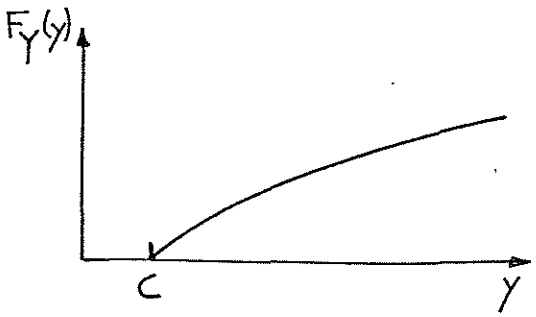
$$\ell n\, c + x \le \ell n\, y$$

$$x \le \ell n\, y - \ell n\, c = \ell n \left(\frac{y}{c}\right)$$

2.54 (continued)

$$F_Y(y) = \int_0^{\ell n \frac{y}{c}} \lambda e^{-\lambda x}\, dx = -e^{-\lambda x}\Big|_0^{\ell n(\frac{y}{c})}$$

$$F_Y(y) = -\left(\frac{y}{c}\right)^{-\lambda} + 1 \qquad y \geq c$$



2.55 $p_X(x) = (1-p)^{x-1}\, p \qquad x = 1, 2, 3, \ldots$

$p_Y(y) = (1-p)^{y-1}\, p \qquad y = 1, 2, 3, \ldots$

$Z = X + Y$

$$p_Z(z) = \sum_{\text{all } y} P[(Y=y) \cap (X=z-y)] \sum p_{X,Y}(z-y,y)$$

$p_{X,Y}(x,y) = p_X(x)p_Y(y)$ if X and Y independent

$p_{X,Y}(x,y) = (1-p)^{x-1}p(1-p)^{y-1}\, p = (1-p)^{x+y-2}\, p^2$ for X and Y positive integers

$p_{X,Y}(z-y,y) = p^2(1-p)^{z-y+y-2} = p^2(1-p)^{z-2}$ for z-y and y positive integers

$$p_Z(z) = \sum_{y=1}^{z-1} (1-p)^{z-2}\, p^2 = (1-p)^{z-2}p^2 \sum_{y=1}^{z-1} 1 = (z-1)(1-p)^{z-2}\, p^2, \quad z = 2,3,\ldots$$

Z is the number of specimens that have to be tested before a total of 2 unsatisfactory ones are found. The probability that Z = z is clearly the probability, $(z-1)(1-p)^{z-2}p$ that in the first z-1 specimens exactly z-2 are unsatisfactory and some of the z-1 is unsatisfactory times p, the probability that the last is unsatisfactory. See Sec. 3.1.

2.56 A = water made available since previous September

R = water released since previous September

A - R = S, water in storage during summer

$\sum$ = surface area = cS where c is a deterministic proportionality constant

2.56 (continued)

$L = K\Sigma$ where K is a random proportionality constant

L is water lost due to evaporation

Assume all factors independent.

$L = K\Sigma = KcS = Kc(A-R) = KcA - KcR$

Assume A, R, K uncorrelated. This may not be reasonable for A and R.

$m_L = E[KcA] - E[KcR]$ from Eq. 2.4.23

$= cE[KA] - cE[KR]$ from Eq. 2.4.21

$E[KA] = E[K]\,E[A]$, $E[KR] = E[K]\,E[R]$ from Eq. 2.4.86

$m_L = c\,m_K m_A - c\,m_K m_R$

$m_L = c\,m_K(m_A - m_R)$

$Var[L] = Var[cKA - cKR]$; note $X = KA$ and $Y = KR$ are correlated, since both depend on K. Either find this correlation and apply Eq. 2.4.82 to X and Y or expand:

$Var[L] = E[(cKA - cKR)^2] - m_L^2$

$= c^2E[K^2A^2 - 2K^2AR + K^2R^2] - m_L^2$

Assuming lack of correlation between K^2 and A^2, etc. (strictly this is not necessarily implied by lack of correlation between K and A although it is by independence of K and A), $E[A^2K^2] = E[A^2]E[K^2]$

$Var[L] = c^2(E[K^2]E[A^2] - 2E[K^2]E[A]E[R] + E[K^2]E[R^2]) - m_L^2$

$= c^2E[K^2]\,(E[A^2] - 2m_Am_R + E[R^2]) - m_L^2$

Replacing $E[K^2]$ by $m_K^2 + \sigma_K^2$, etc., and collecting terms

$= c^2[\sigma_K^2(\sigma_A^2 + \sigma_R^2 + (m_A+m_R)^2 + m_K^2(\sigma_A^2 + \sigma_R^2)]$

Note the influence of neglecting any one of the variances.

2.57 a) $E[Z] = \int_{-\infty}^{\infty} z f_Z(z)dz = \int_0^d \frac{z^2 2\lambda dz}{d(\lambda d+2)} + \int_d^{\infty} \frac{z2\lambda e^{-\lambda(z-d)}dz}{\lambda d+2}$

$$= \frac{2}{\lambda d+2}\left[\frac{\lambda d^2}{3} + d + \frac{1}{\lambda}\right]$$

b) $E[Y] = E[g(z)] = \int_{-\infty}^{\infty} q(z) f_Z(z)dz$

$$= \int_0^d z f_Z(z)dz + \int_d^{d+c} d f_Z(z)dz + \int_{d+c}^{\infty} (z-c) f_Z(z)dz$$

$$= \frac{2}{\lambda d+2}\left[\frac{1}{3}\lambda d^2 + d + \frac{1}{\lambda} e^{-\lambda c}\right]$$

Notice that increasing the capacity c will decrease the expected water released. For infinite capacity the expected release is

$\frac{2}{3} d \left(\frac{\lambda d+3}{\lambda d+2}\right)$ which equals $\frac{2}{3} d$ for large λd and d for small λd. For

fixed d, λd is large if λ is large, i.e., if the upper tail of $f_Z(z)$ falls off very quickly. (In this example one cannot separate out easily the design implications of changing the target draft d since distribution of Z includes the same d as a parameter. In fact the target draft is now set equal to the most likely value or mode of Z.)

2.58 $m_R \approx 1.25\ \sigma n^{1/2}$

$\sigma_R \approx 0.272\ \sigma n^{1/2}$

a) $V_R = \frac{\sigma_R}{m_R} = \frac{0.272\ \sigma n^{1/2}}{1.25\ \sigma n^{1/2}} = \frac{0.272}{1.25} = 0.218$

b) $r_o = m_R + K\sigma_R$ = design value

$K = 2 \quad r_o = m_R + 2\sigma_R$

$r_o = 1.25\ \sigma n^{1/2} + 0.544\ \sigma n^{1/2} = 1.794\ \sigma n^{1/2}$

2.58 (continued)

$$\frac{r_o}{m_R} = \frac{1.794\ \sigma n^{1/2}}{1.25\ \sigma n^{1/2}} = 1.44$$

The design range depends only on the second moment of the annual impoundment.

Design for an infinite lifetime is not possible because this would require an infinite design range.

The design range is not too sensitive to the design lifetime. For instance, for n = 50 (and K = 2) $r_o = 1.794\sigma\sqrt{50} = 12.7\sigma$

for n =100 (and K = 2) $r_o = 1.794\sigma\sqrt{100} = 17.9\sigma$

Design range increases as square root of design lifetime.

2.59 $S_{i+1} = a(S_i + X_{i+1} - d)$ = storage at end of (i+1)th interval

X_{i+1} = inflow during (i+1)th interval

d = target release

$0 \leq a \leq 1$

X's uncorrelated with m_X, σ_X^2.

$S_i = a(S_{i-1} + X_i - d)$

$S_{i-1} = a(S_{i-2} + X_{i-1} - d)$

Substituting S_{i-1} into S_i and then S_{i-2} into the resulting expression for S_i, etc., one gets

$$S_i = a(X_i) - ad + a^2(X_{i-1}) - a^2d + a^3(X_{i-2}) - a^3d + \ldots$$

$$S_i = \sum_{n=1}^{i} [a^{i+1-n}(X_n - d)]$$

$$m_S = E[S_i] = \sum_{n=1}^{i} [a^{i+1-n}(m_{X_n} - d)] = (m_X - d) \sum_{n=1}^{i} (a^{i+1-n})$$

For large i

$$\lim_{i\to\infty} \sum_{n=1}^{i} a^{i+1-n} = \frac{a}{1-a}$$

2.59 (continued)

$$m_S = \frac{a}{1-a}(m_X - d)$$

$$\sigma^2_{S_i} = \sum_{n=1}^{i} \mathrm{Var}[a^{i+1-n}(X_n - d)]$$

$$= \sum_{n=1}^{i} (a^{2(i+1-n)})\sigma^2_X$$

$$\sigma^2_S = \frac{a^2}{1-a^2}\sigma^2_X$$

Now check continuity equation for large i. Taking expectation of both sides:

$$m_S = a(m_S + m_X - d)$$

$$\frac{a}{1-a}(m_X - d) = a(\frac{a}{1-a}(m_X - a) + m_X - d)$$

$$\frac{a}{1-a}(m_X - d) = a(\frac{a}{1-a}(m_X - a) + \frac{1-a}{1-a}(m_X - d)) = a(\frac{1}{1-a}(m_X - d)); \quad \text{O.K.}$$

Variances of both sides are:

$$\sigma^2_S = a^2(\sigma^2_S + \sigma^2_X)$$

$$\frac{a^2}{1-a^2}\sigma^2_X = a^2(\frac{a^2}{1-a^2}\sigma^2_X + \sigma^2_X) = a^2(\frac{a^2}{1-a^2}\sigma^2_X + \frac{1-a^2}{1-a^2}\sigma^2_X) = a^2(\frac{1}{1-a^2}\sigma^2_X); \text{ O.K.}$$

For a = 1, as the design life increases, the uncertainty in the carryover in any year (σ^2_S) becomes infinite. The capacity would have to be exceedingly large to maintain this policy.

b) storage at time (i) + inflow during interval (i+1) - Release during interval (i+1) = storage at time (i+1)

$$S_i + X_{i+1} - Y_{i+1} = S_{i+1}$$

$$S_i + X_{i+1} - Y_{i+1} = a(S_i + X_{i+1} - d)$$

$$Y_{i+1} = S_i + X_{i+1} - aS_i - aX_{i+1} + ad$$

$$Y_{i+1} = (1-a)(X_{i+1} + S_i) + ad$$

2.59 (continued)

Substituting as in part a:

$$Y_{i+1} = (1-a)(X_{i+1} + aX_i - ad + a^2X_{i-1} - a^2d + \ldots) + ad$$

For large i:

$$m_Y = (1-a)\left(\frac{1}{1-a}(m_X - ad)\right) + ad$$

$$m_Y = m_X$$

and

$$\sigma_Y^2 = (1-a)(\sigma_{X+i}^2 + a^2\sigma_{X_i}^2 + a^4\sigma_{X_i-1}^2 + \ldots)$$

$$\sigma_Y^2 = (1-a)\left(\frac{1}{1-a^2}\sigma_X^2\right)$$

$$= \frac{1-a}{(1+a)(1-a)}\sigma_X^2$$

$$\sigma_Y^2 = \left(\frac{1-a}{1+a}\right)\sigma_X^2$$

After many years, the average release equals the average inflow.
This is necessary for the system to be stable.
For a = 0, the storage each year is zero, so the release just equals the input. Thus $\sigma_Y^2 = \sigma_X^2$.
For a = 1, the release always equals the demand: $Y_{i+1} = (1-a)(X_{i+1}+S_i)+ad$
$= d$

Therefore there is no uncertainty in the release, and $\sigma_Y^2 = 0$.

c) $$\rho_{S_i,S_{i+1}} = \frac{Cov[S_i,S_{i+1}]}{\sigma_{S_i}\sigma_{S_{i+1}}} = \frac{Cov[S_i,S_{i+1}]}{\frac{a^2}{(1-a^2)}\sigma_X^2}$$

$$Cov[S_i,S_{i+1}] = E[S_i S_{i+1}] - E[S_i]E[S_{i+1}]$$

$$S_i = a(S_{i-1} + X_i - d)$$

$$S_{i+1} = a(S_i + X_{i+1} - d)$$

$$S_iS_{i+1} = a^2(S_iS_{i-1} + S_{i-1}X_{i+1} - dS_{i-1} + X_iS_i + X_iX_{i+1} - dX_i - dS_i - dX_{i+1} + d^2)$$

2.59 (continued)

$$E[S_iS_{i+1}] - E[S_i]E[S_{i+1}] = a^2E[S_iS_{i-1}] + a^2E[X_iS_i] - a^2m_S^2 - a^2m_Xm_S$$

$$S_i = a(S_{i-1} + X_i - d)$$

$$S_{i-1} = \frac{S_i}{a} - X_i + d$$

$$E[S_iS_{i-1}] + E[X_iS_i] = E[S_i(S_{i-1} + X_i)]$$

$$= E[S_i(\frac{S_i}{a} - X_i + d + X_i)]$$

$$= E[\frac{1}{a}S_i^2 + S_id]$$

$$= \frac{1}{a}[m_S^2 + \sigma_S^2] + dm_S$$

$$E[S_iS_{i+1}] - E[S_i]E[S_{i+1}] = am_S^2 + a\sigma_S^2 + a^2dm_S - a^2m_S^2 - a^2m_Xm_S$$

$$\rho = \frac{am_S^2+a\sigma_S^2+a^2dm_S-a^2m_S^2-a^2m_Xm_S}{\frac{a^2}{1-a^2\sigma_X^2}}$$

After substitution for σ_S and m_S, $\rho = a$.

If $a = 1$, we can predict S_{i+1} perfectly once we know S_i; in fact both are always d.

If $a = 0$, we have no better estimate of S_{i+1} if we know S_i, since they equal the respective (independent) inflows X_{i+1} and X_i.

d) For $a = 1$, $y = d$ is only valid for $d \leq z \leq d+c$.

If the available water is less than z, there will not be enough water to release d.

If the available water is greater than d+c, releasing d will leave the dam overflowing, implying a release greater than d.

2.60 a) Y has uniform distribution $f_Y(y) = \frac{2}{3d}$ $\qquad \frac{1}{2}d \le y \le 2d$

$$u = gy + (v-gd) \qquad y \le d$$

$$u = hy + (v-hd) \qquad y \ge d$$

$$E[U] = \int_{d/2}^{d} \{gy + (v-gd)\}\frac{2}{3d}\,dy + \int_{d}^{2d} \{hy + (v-hd)\}\,\frac{2}{3d}\,dy$$

$$= \frac{2}{3d}\,[3gd^2/8 + (v-gd)d/2 + 3hd^2/2 + (v-hd)d]$$

b) If $f_Y(y)$ is available simply integrate as in part a. If not, find U as function of Z. From Prob. 2.32:

$$y = z \qquad 0 \le z \le d$$
$$y = d \qquad d \le z \le d+c$$
$$y = z-c \qquad d+c \le z$$

$$u = gy + (v-gd) = gz + (v-gd) \qquad 0 \le z \le d$$
$$u = hy + (v-hd) = hd + v - hd = v \qquad d \le z \le d+c$$
$$u = hy + (v-hd) = h(z-c) + (v-hd) \qquad d+c \le z$$

From Prob. 2.32 we have $f_Z(z)$:

$$E[U] = \int_{o}^{d} \{gz + (v-gd)\}\,\frac{2\lambda z\,dz}{d(\lambda d+2)} + \int_{d}^{d+c} \{v\}\,\frac{2\lambda e^{-\lambda(z-d)}dz}{\lambda d+2}$$

$$+ \int_{d+c}^{\infty} \{h(z-c) + v - hd\}\,\frac{2\lambda e^{-\lambda(z-d)}dz}{\lambda d+2}$$

$$E[U] = \frac{2}{\lambda d+2}\,\{-\frac{g\lambda d^2}{6} + \frac{v\lambda d}{2} + v + \frac{h}{\lambda}\,e^{-\lambda c}\}$$

The dependence on g and h is of interest, but in this form it is difficult to deduce the dependence on the target release d, because the distribution of Z has been written in terms of d.

2.61 (f-h) = benefit per unit Y above $\tilde{d}$

(g-f) = benefit per unit Y below $\tilde{d}$ (actually (f-g) per negative unit Y below $\tilde{d}$)

(f-h) (fraction of time above $\tilde{d}$) = (g-f) (fraction of time below $\tilde{d}$)

$$\frac{\text{fraction below}}{\text{fraction above}} = \frac{f-h}{g-f}$$

and fraction above + fraction below = (f-h) + (g-f) = g-h

$\therefore \frac{f-h}{g-h}$ = percentage of time below $\tilde{d} \equiv F_Y(\tilde{d})$

$\frac{g-f}{g-h}$ = percentage of time above $\tilde{d} \equiv 1 - F_Y(\tilde{d})$

Find the expected value of U as a function of d and differentiate with respect to d. Let v_o equal the v-axis intercept of v(d).

$$E[U] = \int_0^d [gy + (v_o+fd-gd)]\, f_Y(y)dy + \int_d^\infty [hy + (v_o+fd-hd)]f_Y(y)dy$$

$$= \int_0^d gyf_Y(y)dy + (v_o+fd-gd)F_Y(d) + \int_d^\infty hyf_Y(y)dy + (v_o+fd-hd)[1-F_Y(d)]$$

Taking derivatives with respect to d, setting equal to zero and solving gives the desired result.

2.62 a) i) Since $e^{uX} = 1 + uX + \frac{1}{2}(uX)^2 + \frac{1}{3!}(uX)^3 + \ldots$

$$M(u) = E[e^{uX}]$$

$$= 1 + uE[X] + \frac{1}{2}u^2E[X^2] + \frac{1}{6}u^3E[X^3] + \ldots$$

$$\frac{dM(u)}{du} = E[X] + uE[X^2] + \frac{1}{2}u^2E[X^3] + \ldots$$

$$\left.\frac{dM(u)}{du}\right|_{u=0} = E[X]$$

2.62 (continued)

ii)

$$\frac{d^2M(u)}{du^2} = E[X^2] + uE[X^3] + \ldots$$

$$\left.\frac{d^2M(u)}{du^2}\right|_{u=0} = E[X^2]$$

b) $e^{iuX} = 1 + iuX + \frac{1}{2}(iuX)^2 + \frac{1}{3!}(iuX)^3 + \ldots$

$\zeta(u) = E[e^{iuX}]$

As above $\frac{d^k\zeta(u)}{du^k} = i^kE[X^k] + ui^{k+1}E[X^{k+1}]$ + other terms involving u

Thus $\left.\frac{d^k\zeta(u)}{du^k}\right|_{u=0} = i^kE[X^k]$

and $E[X^k] = \frac{1}{i^k}\left.\frac{d^k\zeta(u)}{du^k}\right|_{u=0}$

c) $M_Y(u) = E[e^{uY}]$

but $Y = q(X_1,X_2) = X_1 + X_2$

$E[e^{uY}] = E[e^{u(X_1+X_2)}] = E[e^{uX_1}e^{uX_2}]$

Since X_1 and X_2 are independent, the expectation of the product factors into the product of the expectations (this can be demonstrated by substituting into the integral definition of expectation).

$E[e^{uY}] = E[e^{uX_1}]E[e^{uX_2}]$

or $M_Y(u) = M_{X_1}(u)M_{X_2}(u)$

Proof for the characteristic function is similar.

2.62 (continued)

d)

$$M_R(u) = \int_0^\infty e^{ur} 2e^{-2r} dr = 2 \int_0^\infty e^{(u-2)r} dr = \frac{1}{2(u-2)} e^{(u-2)r} \Big|_0^\infty$$

For $u < 2$ the integral converges:

$$M_R(u) = -\frac{2}{(u-2)}$$

$$\frac{dM_R(u)}{du} = 2(u-2)^{-2} \; ; \quad E[X] = \frac{dM_R(u)}{du}\Big|_{u=0} = \frac{1}{2}$$

$$\frac{d^2M_R(u)}{du^2} = -4(u-2)^{-3}; \quad E[X^2] = \frac{d^2M_R(u)}{du^2}\Big|_{u=0} = \frac{1}{2}$$

Similarly, $E[X^3] = \frac{d^3M(u)}{du^3}\Big|_{u=0} = \frac{3}{4}$

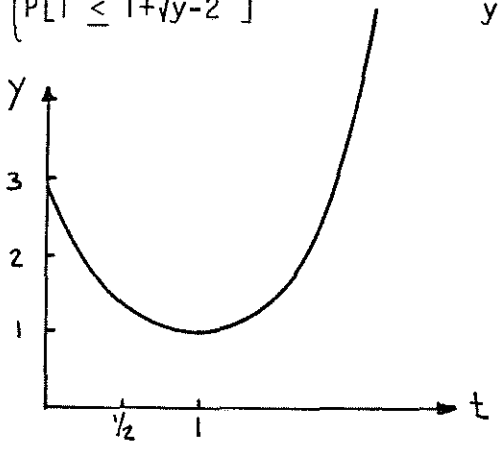
$$E[X^4] = 1\frac{1}{2}$$

2.63 The relationship between Y and T is <u>not</u> one-to-one.

a) $t = 1 \pm \sqrt{y-2}$ $\qquad t \geq 0; \; y \geq 2$

$$F_Y(y) = P[Y \leq y]$$

$$= \begin{cases} P[1-\sqrt{y-2} \leq T \leq 1+\sqrt{y-2}\,] & 2 \leq y \leq 3 \\ P[T \leq 1+\sqrt{y-2}\,] & y \geq 3 \end{cases}$$



2.63 (continued)

$$F_Y(y) = \begin{cases} F_T(1+\sqrt{y-2}\,) - F_T(1-\sqrt{y-2}\,) & 2 \le y \le 3 \\ F_T(1+\sqrt{y-2}) & y \ge 3 \end{cases}$$

In this case

$$F_T(t) = 1 - e^{-3t} \qquad t \ge 0$$

$$F_Y(y) = \begin{cases} e^{-3(1-\sqrt{y-2})} - e^{-3(1+\sqrt{y-2})} & 2 \le y \le 3 \\ 1 - e^{-3(1+\sqrt{y-2})} & y \ge 3 \end{cases}$$

$$f_Y(y) = \begin{cases} 3y^{-1/2}\, e^{-3}[e^{3\sqrt{y-2}} - e^{-3\sqrt{y-2}}] & 2 \le y \le 3 \\ 3y^{-1/2}\, e^{-3} e^{3\sqrt{y-2}} & y \ge 3 \end{cases}$$

b) i) $E[Y] = \int_2^\infty y\, f_Y(y)dy$

ii) $E[Y] = \int_0^\infty g(t)\, f_T(t)dt$

iii) $E[Y] = \int_0^\infty [1-F_Y(y)]dy \overset{\text{here}}{=} \int_2^\infty [1-F_Y(y)]dy$

The last result is sometimes useful when one has the CDF but not the PDF. It holds if Y is restricted to positive values and if $y(1-F_Y(y)) = 0$ at $y = \infty$. It can be proved easily by applying the formula for integration by parts to the definition of the mean. By any of these formulas (the second is easiest to calculate)

$$E[Y] = E[(T-1)^2 + 2] = E[T^2] - 2E[T] + 3 = \frac{2}{\lambda^2} - \frac{2}{\lambda} + 3$$

2.64 Assume for simplicity that the best estimates (means) of the costs (X_1, X_2, X_3, and X_4) are all equal to m, and that the uncertainties in the estimates of the costs are comparable and measurable by σ or $V = \sigma/m$. The variance of the total cost $Y = \Sigma X_i$ is (for equal ρ_{ij})

$$\sigma_Y^2 = 4\sigma^2 + 2 \sum_{i=1}^{4} \sum_{j=i+1}^{4} \rho_{ij}\sigma^2 = \sigma^2(4+12\rho)$$

$$\sigma_Y = \sigma 2\sqrt{1+3\rho}$$

Owing to their dependence on similar but unknown soil conditions, the costs will be positively correlated. Therefore σ_Y is greater than σ. In this sense there is <u>greater</u> uncertainty in the total cost than in the cost of any bridge, i.e., the likelihood that the true total cost is within n dollars of the estimate is less than the likelihood that the cost of any one bridge is within n dollars of its estimate.

On the other hand

$$V_Y = \sigma_Y/m_Y = \sigma\sqrt{1+3\rho}/4m = V\sqrt{1+3\rho}/4$$

Unless $\rho = 1$, V_Y is less than V. In words, the likelihood that true total cost is within p percent of the estimate is greater than the likelihood that the true cost of any one bridge is within p percent of its estimate. The implication of proximity is high correlation. The greater this correlation the higher σ_Y and V_Y, i.e., the <u>less</u> likely it is that, as the engineer predicted, one high cost will be balanced by one low cost, since the correlation implies all will tend to be high or all will tend to be low.

2.65 $C = YP$

$E[YP] = E[Y]E[P] = 600{,}000 = E[C]$

$\text{Var}[YP] = m_y^2\sigma_p^2 + m_p^2\sigma_y^2 + \sigma_p^2\sigma_y^2$

$= 28.625 \times 10^8 = \text{Var}[C]$ exact

Approximate variance (mean is exact):

$\text{Var}[YP] \cong m_y^2\sigma_p^2 + m_p^2\sigma_y^2$

$= (10^{10})(.24) + (36)(10^7)$

$= 28.60 \times 10^8 = \text{Var}[C]$ approximate

Approximate answer is close because product of variances is small compared to other terms. This will happen if either (or both) coefficients of variation is small.

2.66 a) System reliability, $1 - p_s = P[\text{all o.k.}] = P[C_1 \cap C_2 \ldots \cap C_n]$

$$= \prod_{i=1}^{n} P[C_i] = \prod_{i=1}^{n} 1-P[C_i^c] = \prod_{i=1}^{n} (1-p_i) = 1- \sum_{i=1}^{n} p_i + \sum\sum p_i p_j - \cdots$$

$$\cong 1 - \sum_{i=1}^{n} p_i$$

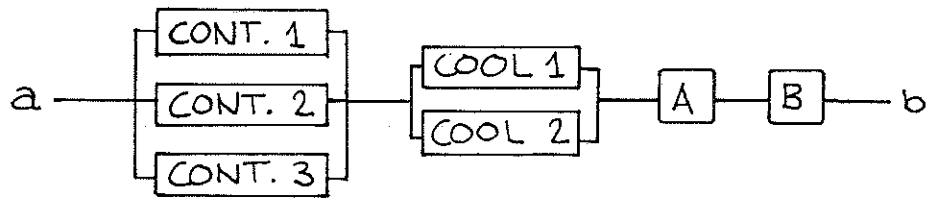
b) System reliability $1 - p_s = 1-P[\text{none o.k.}]$

$$= 1-P[C_1^c \cap C_2^c \ldots \cap C_n^c] = 1 - \prod_{i=1}^{n} P[C_i^c] = 1 - \prod_{i=1}^{n} p_i$$

c) $1-p_s = P[C_1 \cap (C_2 \cup C_3)] = P[C_1]P[C_2 \cup C_3]$

$= P[C_1](1-P[C_2^c \cap C_3^c]) = (1-p_1)(1-p_2p_3)$

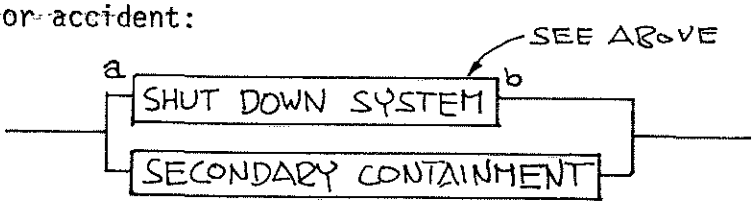
2.66 (continued)
d)



$1-p_{cs}$ = P[controlled shutdown]

$= (1-p_{CONT_1}p_{CONT_2}p_{CONT_3})(1-p_{COOL_1}p_{COOL_2})(1-p_A)(1-p_B)$

Major accident:



Reliability = P[no major accident] = $1-p_{cs}p_{SEC.CONT.}$

e) See part c: $1-p_s = P[C_1](1-P[C_2^c \cap C_3^c])$

$= P[C_1](1-P[C_2^c]P[C_3^c|C_2^c])$

$= (1-p_1)(1-p_2p_3^*)$

2.67 a) $T_n = \sum_{i=1}^{n} L_i = \sum a(B+S_i) = anB + a\sum S_i$

$E[T_n] = E[\sum L_i] = \sum E[L_i] = \sum a(E[B] + E[S]) = anm_B$

$Var[T_n] = Var[\sum L_i] = Var[anB + a\sum S_i]$

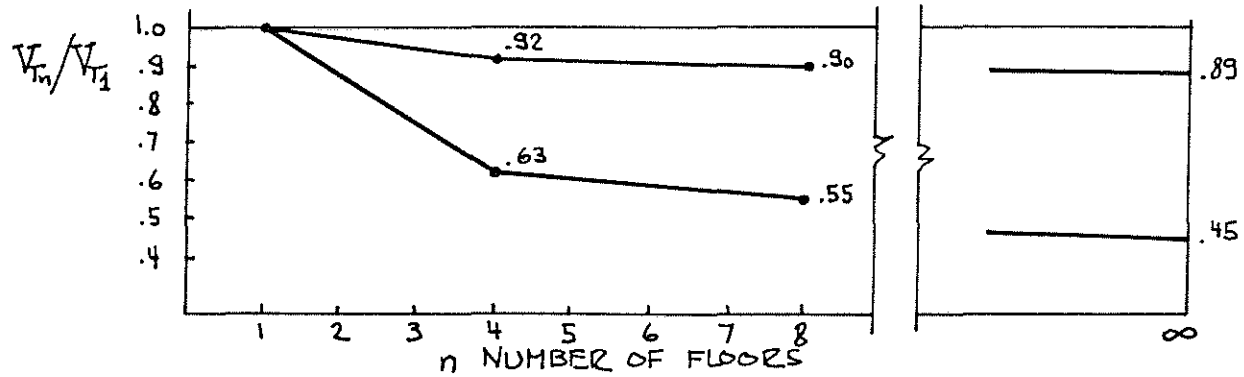
L_i's are correlated, but B and S_i's are not. Therefore

2.67 (continued)

$$\text{Var}[T_n] = a^2n^2\sigma_B^2 + a^2n\sigma_S^2 = a^2n^2(\sigma_B^2 + \frac{1}{n}\sigma_S^2)$$

b) $$V_{T_n} = \sigma_{T_n}/m_{T_n} = an\sqrt{\sigma_B^2 + \sigma_S^2/n}\ /anm_B$$

$$= V_B\sqrt{1 + \sigma_S^2/n\sigma_B^2}$$



In many modern building codes the "live-load reduction factor" is related to the coefficient of variation of the total live load. Clearly the relative degree of building-to-building versus within building floor-to-floor uncertainty must influence the rate by which this factor changes with number of floors.

c) $$\rho_{L_i,L_j} = \text{Cov}[L_i,L_j]/\sigma_i\sigma_j$$

$$\text{Cov}[L_i,L_j] = E[L_iL_j] - E[L_i]E[L_j]$$

$$= E[a(B+S_i)a(B+S_j)] - a^2m_B^2$$

$$= a^2E[B^2+BS_i+BS_j+S_iS_j] - a^2m_B^2$$

2.67 (continued)

$$= a^2(E[B^2]+0+0+0) - a^2m_B^2 = a^2(\sigma_B^2+m_B^2) - a^2m_B^2$$

$$= a^2\sigma_B^2$$

$$\rho_{L_i,L_j} = a^2\sigma_B^2/a^2(\sigma_B^2+\sigma_S^2) = \frac{1}{1+\sigma_S^2/\sigma_B^2}$$

The larger the building-to-building uncertainty relative to the within building floor-to-floor uncertainty the larger correlation between floor loads.

d) $$\rho_{L_i,L_j|B} = \frac{Cov[L_i,L_j|B=b_o]}{\sigma_{L_i|B=b_o}\,\sigma_{L_j|B=b_o}}$$

$$Var[L_i|B=b_o] = E[L_i^2|B=b_o] - E^2[L_i|B=b_o]$$

$$= E[a^2(b_o+S_i)^2] - E^2[a(b_o+S_i)]$$

$$= a^2(b_o^2+2b_oE[S_i]+E[S_i^2])-a^2(b_o+E[S_i])^2 = a^2E[S_i^2]$$

$$= a^2\sigma_S^2$$

$$Cov[L_i,L_j|B=b_o] = E[L_iL_j|B=b_o] - E[L_i|B=b_o]E[L_j|B=b_o]$$

$$= a^2E[(b_o+S_i)(b_o+S_j)] - a^2(b_o+E[S_i])(b_o+E[S_j])$$

$$= a^2E[S_iS_j] = 0 \quad i \neq j$$

$$\rho_{L_i,L_j|B} = 0 \quad i \neq j$$

If the model holds, the implication is that within a given building the floor loads are uncorrelated. Whereas, if one considers <u>all</u> buildings, the floor loads within a building are correlated (part c).

2.67 (continued)

In other words, buildings are typically heavily loaded or lightly loaded (reflecting common usage or age, perhaps), but within a particular building heavily loaded floors do not tend to "stick together." (Any systematic trend in all buildings, e.g., lower floors heavier than upper floors, could be accounted for by making the mean of the S_i's a function of i, without altering the conclusions about correlation, which is associated with <u>deviations</u> from the mean.)

2.68 $Z = Y - X$

$$f_Z(z) = \int_{-\infty}^{\infty} f_Y(z+x)\, f_X(x)dx$$

$f_X(x)$ is zero for $x < 0$; therefore

$$f_Z(z) = \int_{0}^{\infty} f_Y(z+x)\, \lambda e^{-\lambda x}dx$$

$f_Y(y)$ is zero for $y < 0$; therefore $f_Y(z+x)$ is zero for $z+x < 0$:

$$f_Z(z) = \begin{cases} \int_{-z}^{\infty} \nu e^{-\nu(z+x)} \lambda e^{-\lambda x}dx & \text{for } z < 0 \\ \int_{0}^{\infty} \nu e^{-\nu(z+x)} \lambda e^{-\lambda x}dx & \text{for } z > 0 \end{cases}$$

$$f_Z(z) = \begin{cases} \frac{\nu\lambda}{\nu+\lambda} e^{\nu z} & z < 0 \\ \frac{\nu\lambda}{\nu+\lambda} e^{-\lambda z} & z > 0 \end{cases}$$

2.68 (continued)

$$W = kZ$$

$$f_W(w) = \frac{1}{k} f_Z(w/k) = \begin{cases} \frac{\nu\lambda}{k(\nu+\lambda)} e^{\nu w/k} & w < 0 \\ \frac{\nu\lambda}{k(\nu+\lambda)} e^{-\lambda w/k} & w > 0 \end{cases}$$

2.69 a) $h(t)dt = P[\text{failure in } t \text{ to } t+dt | \text{no failure prior to } t]$

$$= P[t \leq T \leq t+dt | T \geq t]$$

$$= \frac{f_T(t)dt}{1-F(t)}$$

b) $\frac{f(t)}{1-F(t)} = h(t)$; let $G(t) = 1-F(t)$

then $f(t) = \frac{dF(t)}{dt} = -\frac{dG(t)}{dt}$

Substituting $\frac{\frac{dG(t)}{dt}}{G(t)} = -h(t)$

Solving for G(t) by integrating both sides

$$\ell n\, G(t) = -\int h(t)dt$$

Our interest is only in $t \geq 0$

$$\ell n\, G(t) - \ell n\, G(0) = -H(t)$$

$$\ell n\, G(t) = -H(t)$$

$$G(t) = e^{-H(t)} \quad \text{or} \quad F(t) = 1 - e^{-H(t)}$$

2.69 (continued)

c)

$$h(t) = \frac{f(t)}{1-F(t)} = \frac{\nu e^{-\nu t}}{1-(1-e^{-\nu t})} = \frac{\nu e^{-\nu t}}{e^{-\nu t}} = \nu$$

d) Let T_1 = time to first event of first kind which would cause failure, with PDF $f_1(t)$ and CDF $F_2(t)$.

Let T_2 = time to first event of second kind which would cause failure, with PDF $f_1(t)$ and CDF $F_2(t)$.

Then $T = \min[T_1, T_2]$ and therefore

$$1 - F_T(t) = P[T > t] = P[(T_1 > t) \text{ and } (T_2 > t)]$$

$$= [1-F_1(t)][1-F_2(t)]$$

$$f_T(t) = \frac{dF_T(t)}{dt} = f_1(t) + f_2(t) - F_1(t)f_2(t) - F_2(t)f_1(t)$$

Therefore

$$h(t) = \frac{f_T(t)}{1-F_T(t)} = \frac{f_1(t)+f_2(t)-F_1(t)f_2(t)-F_2(t)f_1(t)}{[1-F_1(t)][1-F_2(t)]}$$

$$= \frac{f_1(t)}{1-F_1(t)} + \frac{f_2(t)}{1-F_2(t)}$$

$$= h_1(t) + h_2(t)$$

Alternatively one can argue directly in terms of $h(t)dt$ and proper conditional probability (i.e., the conditional probability that T_1 or T_2 lies in the interval given that T_1 and T_2 are greater than t.)

Here $h_1(t) = \nu$ and $h_2(t) = \frac{\nu}{10} t$, therefore

$$h(t) = \nu(1+t/10)$$
$$H(t) = \nu(t+t^2/20)$$

2.69 (continued)

$$F_T(t) = 1-\exp[-\nu t(1+t/20)]$$

$$R(20) = 1-F_T(20) = \exp[-40\nu]$$

2.70 a) $m_Y = m_1 + m_2$; $\quad \sigma_Y^2 = \sigma_1^2 + \sigma_2^2$

b) $E[YX_2] = E[(X_1+X_2)X_2] = E[X_1X_2] + E[X_2^2] = m_1m_2 + (m_2^2 + \sigma_2^2)$

$$\rho_{YX_2} = \mathrm{Cov}[Y, X_2]/\sigma_Y\sigma_2 = (E[YX_2] - m_Ym_2)/\sigma_Y\sigma_2$$

$$= (m_1m_2 + m_2^2 + \sigma_2^2 - m_1m_2 - m_2^2)/\sigma_2^2\sqrt{1+(\sigma_1/\sigma_2)^2}$$

$$= 1/\sqrt{1 + (\sigma_1/\sigma_2)^2}$$

The correlation coefficient between the total deformation Y and the deformation of the second floor X_2 is independent of the means, is close to 1 if σ_2 dominates σ_1, and is close to zero if σ_2 is negligible compared to σ_1.

c) With X_1 and X_2 correlated, $\mathrm{Cov}[X_1, X_2] = \rho\sigma_1\sigma_2$

$$E[X_1X_2] = \mathrm{Cov}[X_1, X_2] + m_1m_2 = \rho\sigma_1\sigma_2 + m_1m_2$$

$$E[YX_2] = \rho\sigma_1\sigma_2 + m_1m_2 + (m_2^2 + \sigma_2^2)$$

$$\rho_{YX_2} = (1 + \rho\sigma_1/\sigma_2)/\sqrt{1+(\sigma_1/\sigma_2)^2}$$

The correlation between Y and X_2 is higher than in b, for any σ_1/σ_2 ratio.

2.71 $m_{X_1} = 2 = k_1/\lambda_1$; $\sigma^2_{X_1} = 2 = k_1/\lambda_1^2$ so $\lambda_1 = 1$; $k_1 = 2$

$m_{X_2} = 3$; $\sigma^2_{X_2} = 3$ so $\lambda_2 = 1$; $k_2 = 3$

$$f_Y(y) = \int_0^y f_{X_1}(x)\, f_{X_2}(y-x)dx$$

$$= \int_0^y xe^{-x}\, \frac{1}{2!}\, (y-x)^2\, e^{-(y-x)}dx$$

$$= \frac{e^{-y}}{2!} \int_0^y x(y-x)^2dx$$

$$= \frac{1}{4!}\, y^4e^{-y}$$

which is of the same form but with $\lambda = 1$ and $k = 5$. See section 3.2.3.

2.72 Let R = the resistance

$$p_R(r) = \begin{cases} 0.25 & \text{for } r=100 \\ 0.50 & \text{for } r=120 \\ 0.25 & \text{for } r=140 \end{cases}$$

Reliability = P[R > X]

$$= P[R=100]P[X<100] + P[R=120]P[X<120] + P[R=140]P[X<140]$$

$$= \frac{1}{4}\, F_X(100) + \frac{1}{2}\, F_X(120) + \frac{1}{4}\, F_X(140)$$

$$= 0.25(0.86) + 0.50(0.91) + 0.25(0.94)$$

$$= 0.22 + 0.45 + 0.23 = 0.90$$

2.73 $E[R] = kE[X_i]$

$E[X_i] = 0P[X_i=0] + 1P[X_i=1] = P[X_i=1]$

$P[X_i=1]$ = P[elevator stops at i^{th} floor]

= 1 - P[elevator does <u>not</u> stop at i^{th} floor]

= 1 - P[<u>all</u> n people choose a floor other than the i^{th}]

$= 1 - (1 - \frac{1}{k})^n$

Thus $E[R] = k[1 - (1 - \frac{1}{k})^n]$

For moderate n/k, $E[R] \cong k(1 - e^{-n/k})$. For small n/k, $E[R] \cong n$.

2.74 Either find distribution of maximum velocity and then the distribution of the corresponding force, or find the distribution of peak annual force and then the distribution of the maximum force. The former approach is, letting Z equal $\max[X_i]$,

$$F_Z(z) = F^n_{X_i}(z) = e^{-n(z/u)^{-k}} = e^{-(z/u_o)^{-k}} \qquad z > 0$$

in which $u_o = u/n^{-1/k}$. Then

$$Y = cZ^2$$

$$F_Y(y) = P[Y \leq y] = P[cZ^2 \leq y] = P[Z \leq \sqrt{y/c}\,]$$

$$= F_Z(\sqrt{y/c}) = \exp[-(\sqrt{y}/u_o\sqrt{c})^{-k}$$

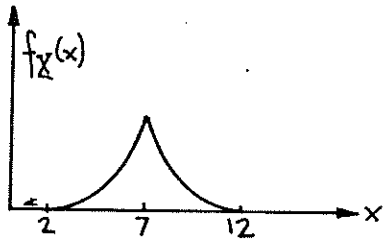
$$= \exp[-y/cu_o^2]^{-k_o} \qquad y > 0$$

in which $k_o = k/2$. In short Y has a distribution of the same form as X but with $cu_o^2 = cu^2n^{2/k}$ in place of u and $k_o = k/2$ in place of k. This form is discussed in Section 3.3.3.

2.75 a) Normalize PDF to find k:

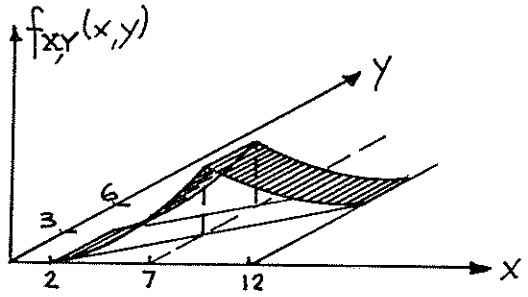
$$1 = \int_{-\infty}^{\infty} f_X(x)dx = k\int_2^7 (x-2)^2dx + k\int_7^{12} (12-x)^2dx = k\ 250/3$$

$k = 3/250$



b) $f_{X,Y}(x,y) = f_{Y|X}(y,x)f_X(x)$

$$= \begin{cases} \frac{1}{3} \cdot \frac{3}{250}(x-2)^2 & 2 \le x \le 7;\ \frac{1}{2}x-1 \le y \le \frac{1}{2}x+2 \\ \frac{1}{3} \cdot \frac{3}{250}(12-x)^2 & 7 \le x \le 12;\ \frac{1}{2}x-1 \le y \le \frac{1}{2}x+2 \\ 0 & \text{elsewhere} \end{cases}$$

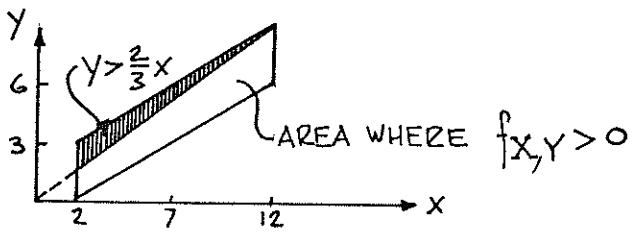


c) Rate = Y/X

$P[Y/X \ge \frac{2}{3}] = P[Y \ge \frac{2}{3}X]$ = volume over area shown

$$= \int_2^7 dx \int_{\frac{2}{3}x}^{x/2+2} dy\ \frac{(x-2)^2}{250} + \int_7^{12} dx \int_{\frac{2}{3}x}^{x/2+2} dy\ \frac{(12-x)^2}{250}$$

$= 0.27$



CHAPTER 3

3.1 Let Y = no. of successes:

all successes

$$p_Y(m) = \binom{m}{m} p^m (1-p)^0 = p^m$$

all failures

$$p_Y(0) = \binom{m}{0} p^0 (1-p)^m = (1-p)^m$$

P[at least two] = $P[Y \geq 2] = 1 - F_Y(1) = 1 - 0.924 = 0.076$;

P[two or more] = $P[Y \geq 2] = 0.076$; P[two or fewer] = $P[Y \leq 2]$

$= F_Y(2) = 0.997$

3.2 Independent Bernoulli trials with p = P[success] = 3/3+1 = 0.75

a) SSSS, SSSF, SSFS, SSFF, SFSS, SFSF, SFFS, SFFF

FSSS, FSSF, FSFS, FSFF, FFSS, FFSF, FFFS, FFFF

b) 6 contain two successes

c) $(0.75)^2(0:25)^2 = (\frac{9}{16})(\frac{1}{16}) = \frac{9}{256}$

d) $6(\frac{9}{256}) = \frac{27}{128}$

$p_Y(2) = \binom{4}{2} (0.75)^2(0.25)^2 = \frac{27}{128}$

3.3 a) Bernoulli trials with probability of success p = 0.2.

Let N = trial number of first passenger going to 6. $p_N(n) = p(1-p)^{n-1}$

$n = 1,2,...$

Let T = number of arrivals before first passenger going to 6.

$p_T(t) = (1-p)^t p \quad t = 0,1,2,...$

3.3 (continued)

b) The car will be full if at least five of the next seven arrivals will have destination 4. Bernoulli trials with p = 0.2. Let Y = no. of successes.

$$P[Y \geq 5] = \sum_{y=5}^{7} p_Y(y) = \sum_{y=5}^{7} \binom{7}{y}(0.2)^y(0.8)^{7-y}$$

$$= 0.00444 + 0.00036 + 0.00001$$

$$= 0.00481$$

c) Bernoulli trials with probability of success p = 0.6 = P[arrival has destination 4, 5, or 6]. Let X = no. of successes The car will be full if $X \geq 5$.

$$P[X \geq 5] = \sum_{y=5}^{7} \binom{7}{y}(0.6)^y(0.4)^{7-y}$$

$$= 0.2615 + 0.1305 + 0.028$$

$$= 0.42$$

3.4 Bernoulli trials with probability of success

p = P[making left turn] = 0.3

X = number of left turns out of six successive vehicles

$$p_X(x) = \binom{6}{x}(0.3)^x(0.7)^{6-x} \qquad \text{(binomial)}$$

The probability that four of six cars will turn left = P[X=4] =

$$p_X(4) = \binom{6}{4}(0.3)^4(0.7)^2 = (15)(.0081)(.49) = 0.0595$$

The probability that four or more will turn left = $P[X \geq 4]$ =

$$\sum_{x=4}^{6} \binom{6}{x}(0.3)^x(0.7)^{6-x}$$

$$= 0.0595 + (6)(.0081)(0.3)(0.7) + (.0081)(.09)$$

$$= 0.0705$$

3.5 Bernoulli trials with $p = 1/N$. Let Y = no. of floods equal to or exceeding N-year flood.

a) $p_Y(1) = \binom{50}{1}(0.02)^1(0.98)^{49}$

$= .372$

b) $p_Y(3) = \binom{50}{3}(0.02)^3(0.98)^{47}$

$= 0.0605$

c) $P[Y \geq 1] = 1 - p_Y(0) = 1 - (0.98)^{50} = 1 - 0.364 = 0.636$

d) For each system, each of 50 years is a Bernoulli trial:

$P[\text{at least one failure}] = 1 - P[\text{no failures}] = 1 - (.998)^{50} \cong 0.0952 \approx 0.10$

$= 1 - (1 - 0.002)^{50} \cong 1 - 1 + 50(0.002) = 0.1$

Each system represents one of 20 Bernoulli trials with probability of failure 0.1. Let N = no. of failing systems.

$p_N(n) = \binom{20}{n}(0.1)^n(0.9)^{20-n}$

e) Let Y equal no. of floods $\geq$ 50 year flood in 10 years.

$p_Y(2) = \binom{10}{2}(.02)^2(.98)^8 = 45(.0004)(.85)$

$= 0.0153$

This is either an observation of a very unlikely event or the original estimate (0.02) of the annual probability of exceeding this flood magnitude was too small.

3.6 $f_M(m) = \lambda e^{-\lambda m}$ (exponential)

with $\lambda = 2.35$

$f_M(m) = 2.35\ e^{-2.35m}$ $\quad m \geq 0$

$F_M(m) = 1 - e^{-2.35m}$ $\quad m \geq 0$

3.6 (continued)

$$P[M \geq 6.3] = 1 - F_M(6.3) = e^{-(2.35)(6.3)} = e^{-14.8}$$

$$= 3.72 \times 10^{-7}$$

This is the fraction of all earthquakes greater than magnitude 0 which exceed 6.3.

The conditional probability of an earthquake greater than 7.7 given an event greater than 6.1 is

$$P[M \geq 7.7 | M \geq 6.1] = P[(M \geq 7.7) \cap (M \geq 6.1)]/P[M \geq 6.1]$$

$$= P[M \geq 7.7]/P[M \geq 6.1] = [1-F_M(7.7)]/[1-F_M(6.1)]$$

$$= e^{-2.35(7.7)}/e^{-2.35(6.1)}$$

$$= e^{-2.35(1.6)} = 0.0235$$

Therefore the average fraction of all events ≥ 6.1 which are ≥ 7.7 is 0.0235, and the average number of events per year ≥ 7.7 is (1)(0.0235) or 0.0235. Since this number is much less than one, the probability of two or more such events is negligible, and the average number equals the probability of 1, i.e., the annual probability of an occurrence is 0.0235. If this number were not small, additional assumptions (e.g., Poisson arrivals of events ≥ 6.1 and random selection of events ≥ 7.7; see pg. 240) would be necessary to pass from average number to the probability of 1 (or 1 or more) events per year.

3.7 $$P[X \geq 20{,}000] = \int_{20{,}000}^{\infty} 1/6500\ e^{-(x-5500)/6500} dx$$

$$= e^{5500/6500}\ e^{-20{,}000/6500}$$

$$= 0.107$$

3.8 Individual passengers represent Bernoulli trials.

a) Let X = no. transferring at B to subway.

$$p_X(x) = \binom{n}{x} p^x (1-p)^{n-x} \qquad x = 0, 1, \ldots n.$$

b) Let Y = no. leaving at C for Q.

Given x passengers get on subway, each is a Bernoulli trial with probability of success q. Therefore:

$$p_{Y|X}(y,x) = \binom{x}{y} q^y (1-p)^{x-y} \qquad y = 0, 1, \ldots x.$$

The joint distribution is the product of $p_{Y|X}$ and p_X. It is as given for x=0, 1,...n and u = 0, 1, ...x; and it is zero elsewhere. Y cannot exceed X.

c) (i) This is probability $p_{X,Y}(x,x)$ =

$$\frac{n!}{(n-x)!x!} (pq)^x (1-p)^{n-x} \qquad x = 0, 1, \ldots n$$

(ii) This is the probability that none of n want to go beyond B (i.e., that X = 0) plus the probability (from i) that all x get off at Q, for all x.

$$p(x = 0) + \sum_{x=1}^{n} p(x \neq 0 \cap y = x)$$

$$\binom{n}{0} p^0 (1-p)^n + \sum_{x=1}^{n} \frac{n!}{(n-x)!x!} (pq)^x (1-p)^{n-x}$$

$$\sum_{x=0}^{n} \binom{n}{x} (pq)^x (1-p)^{n-x}$$

3.9 $G(t+\Delta t) = P[T \geq t+\Delta t] = P[\text{the molecule is still in the tank at time } t+\Delta t]$

$= P[\text{still in tank at } t]P[\text{it does not leave in t to } t+\Delta t]$

$= G(t)(1-k\Delta t)$

3.9 (continued)

$$\frac{G(t+\Delta t) - G(t)}{\Delta t} = -G(t)k$$

Letting Δt go to zero:

$$\frac{dG(t)}{dt} = -G(t)k$$

Solution of the differential equation with boundary condition $G(0) = 1$ is

$$\therefore \quad G(t) = e^{-kt} \qquad t \geq 0$$

$$f_T(t) = \frac{dF(t)}{dt} = \frac{d}{dt}(1-G(t)) = ke^{-kt} \qquad t \geq 0$$

The probability of leaving in Δt is $k\Delta t = \frac{\text{no. leaving in } \Delta t}{\text{no. in tank}}$

$$= \frac{m\Delta t}{n}$$

Therefore $k = m/n$.

For "complete turbulence", $E[T] = 1/k = n/m$, $\sigma_T = 1/k = n/m$, and $V_T = \sigma_T/m_T = 1$. For both uniform flow and complete mixing the mean time is n/m. In one case, however, $V = 0$, since in the uniform flow $\sigma = 0$. Therefore the coefficient of variation of residence time V would be a good measure, with $V = 0$ for uniform flow, and $V = 1$ for complete turbulence.

3.10 Let X = no. of accidents in 10,000 trials. X is approximately Poisson with $\nu = 5$.

$$P[2 \leq X \leq 8] = \sum_{2}^{8} p_X(x) = \sum_{2}^{8} \frac{5^x e^{-5}}{x!}$$

$$= 0.892$$

For $\nu = 5$

$$P[X \geq 9] = 1 - F_X(8) = 1 - 0.932 = 0.068$$

This seems to be a "not unlikely" event with $\nu = 5$; therefore one cannot "conclude" that ν has increased.

3.11 a) Let X = largest loss of strength in any member; it is the largest of the (random number of) random crack strength losses. Assuming independence of loss and number:

$$P[X \leq x] = \sum_{n=0}^{\infty} P[\text{all n cracks have loss} \leq x]P[\text{there are n cracks}]$$

$$= \sum_{n=0}^{\infty} [F_Y(x)]^n p_N(n)$$

in which $F_Y(x)$ is the CDF of the loss in any crack. Substitution and summing will yield the answer. Alternatively we can consider this a problem of random selection (page 240) in which the probability is p_x that any crack has loss <u>greater</u> than x, or $p_x = 1 - F_Y(x) = e^{-\lambda x}$. Then

$$F_X(x) = P[X \leq x] = P[\text{there are } \underline{\text{no}} \text{ occurrences of cracks with loss} \geq x]$$

Cracks with loss greater than x are Poisson distributed with mean νp_x. Therefore

$$F_X(x) = e^{-\nu p_x} = e^{-\nu(1-F_Y(x))} = \exp[-\nu e^{-\lambda x}] \qquad x \geq 0$$

3.11 (continued)

with $\nu = 2x20 = 40$ and $\lambda = 1(\text{kip})^{-1}$. The remaining strength is $Z = 100 - X$.

$$F_Z(z) = P[100 - X \leq z] = P[X \geq 100-z]$$

$$= 1 - \exp[-\nu e^{-\lambda(100-z)}] = 1 - \exp[-40e^{-(100-z)}] \qquad z \leq 100$$

Note this is a mixed distribution; there is a small but finite probability that Z still equals exactly 100 kips (this occurs if there are no cracks at all). Note too that the distribution suggests a small but finite probability that the strength is negative. Both these probabilities are negligible.

b) $P[\text{failure}] = P[Z \leq 50] = 1 - \exp[-40e^{-(100-50)}]$

$$\cong 40e^{-50} \quad \text{(negligible)}$$

$$P[Z \leq 90] \cong 40e^{-10} = 0.002$$

c) Resistance, R, of the structure is the resistance of the weakest of 10 members, each with the distribution of Z above, assuming each has a unit force under a unit load:

$$1 - F_R(r) = P[\text{all } 10 \geq r] = (1 - F_Z(r))^{10} = \exp[-10\nu e^{-\lambda(100-r)}] \qquad r \leq 100$$

Note the additional members play the same role as additional length.

3.12 $P[A] = P[\text{lane is inadequate}] = P[\text{waiting time to arrival of 4th left-turning car is less than t}]$

$= P[X \leq t]$ in which t is the number of seconds required for one cycle.

The average rate of arrival of left turning vehicles is (0.3)(6) = 1.8/cycle or $\lambda = 1.8/t$ per sec. Since the arrivals are Poisson X is gamma with parameters $k = 4$ and $\lambda = 1.8/t$.

Use Table A.2 where $1-F_X(t) = 1-F(y)$; enter table with $\nu = 2k = 8$ and $\chi^2 = 2\lambda y = 2(1.8/t)t = 3.6$.

Find $1-F_X(t) = 0.89$. Therefore $P[A] = F_X(t) = 1-0.89 = 0.11$.

3.13 Assuming no failure at previous cycle:

$$P[\text{failure}] = P[X \geq 7] \text{ or } P[T_7 \leq 40]$$

in which X is the number of arriving vehicles in 40 seconds, and T_7 is the time of arrival of the seventh car. X is Poisson with mean $\nu = (400)(40)/(60)(60) = 4.4$, or T_7 is gamma distributed with $\lambda = 400/3600$ per second, and $k = 7$. From tables $P[X \geq 7] = 0.156$.

Let X_1 and X_2 equal the number of arrivals in the first and second cycles. Assuming no failure at preceding cycle, the probability that the next two cycles fail is the probability that $X_1 \geq 7$ and $X_1 + X_2 \geq 13$. There are several ways to calculate this, e.g.,

$$P[\text{two cycles fail}] = \sum_{x=7}^{\infty} P[X_2 \geq 13-x]P[X_1=x]$$

$$= (0.28)(0.078) + (0.45)(0.043) + (0.64)(0.021)$$
$$+ (0.82)(0.009) + (0.93)(0.004) + (0.99)(10^{-3}) + (1)(10^{-4})$$
$$= 0.067$$

$$P[\text{second fails} \mid \text{first fails}] = P[\text{two cycles fail}]/P[\text{first fails}]$$

$$= (0.067)/(0.156) = 0.43$$

This suggests that "failures" are quite likely to be self-perpetuating. The model could be improved in many ways; it does not accurately reflect acceleration times, etc.

3.14 $m_X = \lambda v \qquad \sigma_X^2 = \lambda v$

$$V_X = \frac{\sigma_X}{m_X} = \frac{\sqrt{\lambda v}}{\lambda v} = \frac{1}{\sqrt{\lambda}} \frac{1}{\sqrt{v}}$$

Note that this coefficient reduces with volume. For the normal distribution, 0.683 represents an interval of $\pm 1\sigma_X$ about the mean, equating km_X and $1\sigma_X$:

$$km_X = 1\sigma_X$$

$$k = \sigma_X/m_X = V_X = 1/\sqrt{\lambda v}$$

3.14 (continued)

or for given k and λ choose $\nu = 1/\lambda k^2$.

λ is the average number of spheres per unit volume.

3.15 The probability that any earthquake causes site intensity $\geq y$ is $p = 1-F_X(y)$. Therefore, by the random selection model (p. 240) earthquakes causing site intensity larger than y are Poisson arrivals with average rate $p\nu$ (assuming independence of earthquake sizes and occurrences). The probability that Y, the maximum site intensity in t years, is less than y is the probability that number of occurrences exceeding y is zero.

Let N_y = number of occurrences greater than y in time t. N_y is Poisson with mean $p\nu t$

$$F_Y(y) = P[Y \leq y] = P[N_y = 0] = p_{N_y}(0) = e^{-p\nu t}$$

$$= \exp(-\nu t[1 - F_X(y)])$$

It is assumed earthquakes with magnitude $\leq m_o$ cannot cause a site intensity $\geq y$.

3.16 a) $1 - F_X(y) = ce^{-\lambda y}$ $\qquad x \geq x_o = c_1 + c_2 m_o - c_3 \ell nd$

From problem 3.15

$$F_Y(y) = \exp[-\nu t c e^{-\lambda y}] \qquad y \geq x_o$$

Notice that this distribution has the same form as the Type I extreme value distribution, at least for $y \geq x_o$. Below x_o the form of F_X and F_Y is more complicated (Problem 2.54).

b) Expanding the exponential term and ignoring terms of order p^2,

$$p = 1-F_Y(y_p) = 1-\exp[-\nu t_o c e^{-\lambda y_p}] \cong 1-(1-\nu t_o c e^{-\lambda y_p})$$

$$\cong \nu t_o c e^{-\lambda y_p}$$

Solving for y_p

3.16 (continued)

$$y_p \cong \frac{1}{\lambda} \ell n(p/\nu t_o c)$$

c) The "n-year intensity", y_n, is exceeded with probability $p = 1/n$ in time $t_o = 1$ year. Substituting into (b):

$$y_n \cong \frac{1}{\lambda} \ell n(\nu c n)$$

3.17 a) Let $p = 1-p_X(0)$ = probability of 1 or more hurricanes in any year. Then Y is the number of successes in n trials

$$p_Y(y) = \binom{n}{y} p^y (1 - p)^{n-y}$$

b) $p = 1 - p_X(0) = 1 - e^{-0.2} = 0.182$

$E[Y] = np = 30(0.182) = 5.46$

$Var[Y] = np(1-p) = 30(0.182)(0.818) = 4.46$

3.18 a) $F_X(5) = (0.5)^{10} + 10(0.5)^{10} + 45(0.5)^{10} + 120(0.5)^{10} + 210(0.5)^{10}$

$+ 252(0.5)^{10}$

$= 0.622$

$m_X = m_Y = np = 5; \quad \sigma_X^2 = \sigma_Y^2 = np(1-p) = 2.5$

$$F_Y(5) = F_U\left(\frac{5-m}{\sigma}\right) = F_U\left(\frac{5-5}{1.58}\right) = F_U(0) = 0.5$$

b) $$F_Y(5.5) = F_U\left(\frac{5.5-5}{1.58}\right) = F_U(0.317) = 0.624$$

3.19 Normal

$$P[X \geq 400] = 1-F_X(400)=1-F_U\left(\frac{400-m}{\sigma}\right) = 1-F_U\left(\frac{143.3}{191}\right)$$

$$= 1-0.774 = 0.226$$

Lognormal

$$P[X \geq 400] = 1-F_U\left(\frac{\ell n(400)-\ell n\check{m}_y}{\sigma_{\ell ny}}\right)$$

$$= 1-F_U\left(\frac{6-5.228}{0.840}\right) = 1-F_U(.92)$$

$$= 1-0.821 = 0.179$$

Gamma

$$P[X \geq 400] = 1 - \frac{\Gamma(k,\lambda x)}{\Gamma(k)} = 1 - \frac{\Gamma(1.727,\ 2.704)}{\Gamma(1.727)}$$

$$= 1 - \frac{0.71}{0.914} = 1 - 0.78 = 0.22$$

or from Table A.2, enter with

$\nu = 2\ (1.727) = 3.45$ and $\chi^2 = 2(0.00676)(400) = 5.33$

$1 - F(400) = 0.21$

3.20

a) $E[\frac{1}{X^r}] = \frac{1}{m_X^r} E[\frac{1}{(1+V_X U)^r}] = \frac{1}{m_X^r} E[1 - \frac{r}{1!} V_X U + \frac{r(r+1)}{2!} V_X^2 U^2 + \ldots]$

Substitute $E[U^n] = 0$ for n odd

$= \mu_X^{(n)}/\sigma_X^n$ for n even (Eq. 3.3.11)

b) Substituting $m_X = 1$ and $V_X = 0.2$ and following the criterion yields

$E[1/X] = 1 + 0.04 + \ldots \cong 1.0$

$Var[1/X] = E[1/X^2] - E[1/X]^2 \cong 1.12 - 1 = 0.12$

Substituting $m_X = 1$ and $V_X = 0.5$ yields

$E[\frac{1}{X}] = 1 + 0.25 + 0.1875 + 0.234 + 0.410 + \ldots$

i.e., the series does not converge.

In fact the series is divergent for all $V_X > 0$ as can be verified by the ratio test. For the normal distribution $E[1/X^r] = \infty$. The "convergence in the first example is false, higher order terms will begin to grow.

3.21 a) (i) $E[X(m)] = E[Y] \frac{1}{p} \sum_{n=o}^{m} \sin(m-n)$

(ii) $E[Y] = 0$; $E[X(m)] = 0$ (Roughly, the pulses are as likely to be positive as negative, therefore the response is as likely to be positive as negative.)

b) (i) $Var[X(m)] = Var[Y] \frac{1}{p^2} \sum \sin^2(m-n)$

(ii) $Var[Y] = 1$; $Var[X(m)] = \frac{1}{p^2} \sum \sin^2(m-n)$

c) (i) Normal, by the central limit theorem.

3.21 (continued)

c)(ii) limits are $\pm 2.0\,\sigma_X = \pm 2.0\sqrt{\frac{1}{p^2}\sum \sin^2(m-n)}$

d) In general if X has distribution $f_X(x)$ and $Z = |X|$

$$F_Z(z) = P[Z \le z] = P[-z \le X \le z]$$

$$= F_X(z) - F_X(-z)$$

$$f_Z(z) = \frac{d}{dz} F_Z(z) = f_X(z) - f_X(-z)\frac{d(-z)}{dz}$$

$$= f_X(z) + f_X(-z)$$

In part c X is $N(0, \sigma_X^2)$ which is an even function:

$$f_Z(z) = 2f_X(z) = 2\,\frac{1}{\sqrt{2\pi}\sigma_X}\,\exp[-\tfrac{1}{2}(z/\sigma_X)^2] \qquad z \ge 0$$

in which $\sigma_X^2 = \frac{1}{p^2}\sum \sin^2(m-n)$.

3.22 a) $f_T(2.5) = f_U(-2.5)/0.2 = 0.087$

b) $P[T \le 2.5] = F_U(-2.5) = 0.00621$

c) $P[|T-m_T| \le 0.2] = P[-0.2 \le T-m_T \le 0.2]$

$$= F_U(1) - F_U(-1) = 0.8413 - (1-0.8413) = 0.6826$$

d) $P[|T-m_T| \le 0.2 \mid T \ge 2] = \frac{P[(2.8 \le T \le 3.2) \cap (T > 2)]}{P[T > 2]}$

$$= P[2.8 \le T \le 3.2]/P[T > 2] = 0.6826/0.9999 = 0.6826$$

3.23 a) $E[\Delta] = QE[f_1] - QE[f_2] = Qm - Qm = 0$

b) $\sigma_\Delta^2 = Q^2\sigma^2 + Q^2\sigma^2 - 2Q^2\rho\sigma^2$

$\sigma_\Delta^2 = 2Q^2\sigma^2(1 - \rho)$

c) If f_1 and f_2 are jointly normal, Δ is normally distributed.

$\sigma_\Delta = (1000)(0.002)\sqrt{2}\ \sqrt{0.1} = 0.895$ in.

(i) $P[|\Delta| \geq 2] = 1 - [F_U(\frac{2}{0.895}) - F_U(\frac{-2}{0.895})]$

$= 1 - 0.986 + 0.014$

$= 0.028$

(ii) $E[|\Delta|] = \int_{-\infty}^{\infty} |\Delta|\ f_\Delta(\Delta)d\Delta = 2\int_0^{\infty} \Delta \frac{1}{\sqrt{2\pi}\sigma_\Delta} e^{-\frac{1}{2}(\Delta/\sigma_\Delta)^2} d\Delta$

$= 2\sigma_\Delta/\sqrt{2\pi} = 0.8\sigma_\Delta = 0.72$ in.

d) (i) $\sigma_\Delta = (1000)(0.002)\sqrt{2}\ \sqrt{.9} = 2.69$ in.

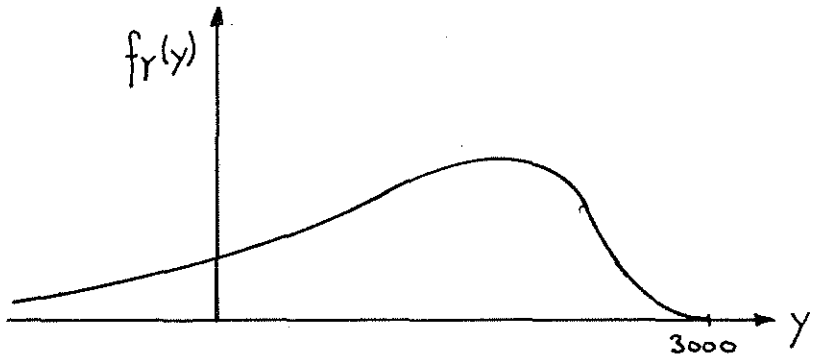
$P[|\Delta| \geq 2] = 0.458$

(ii) $E[|\Delta|] = 2.15$ in.

Note that owing to symmetry the expected value of the differential settlement is zero (part (a)), but that the more meaningful number, the expected value of the <u>absolute value</u> of the differential settlement, is simply proportional to σ_Δ. The less homogeneous the soil the greater the expected absolute value of this settlement.

3.24 $$f_Y(y) = \frac{1}{100} f_X\left(\frac{3000-y}{100}\right)$$

$$= \frac{100}{(3000-y)\sqrt{2\pi}\,\sigma_{\ell n\, X}} \exp\left\{-\frac{1}{2}\left[\frac{1}{\sigma_{\ell n X}} \ell n\left(\frac{3000-y}{100}\right)\right]^2\right\} \qquad y \le 3000$$



Note that Y can be negative (i.e., uplift).

3.25 $$\sigma_{\ell nA} \cong V_A = 0.05$$

$$\breve{m}_A \cong m_A\left(1 - \frac{1}{2}\sigma^2_{\ell nA}\right) = 1(1-0.00125) = 0.999 \text{ in.}$$

F is the product of lognormally distributed variables and is therefore also lognormally distributed with

$$\breve{m}_F = \breve{m}_A\, \breve{m}_S = 39.95 \text{ kips} \cong 40 \text{ kips}$$

$$\sigma^2_{\ell nF} = \sigma^2_{\ell nA} + \sigma^2_{\ell nF} = 0.0125 \text{ or } \sigma_{\ell nF} = 0.112$$

$$P[F \le 35] = F_U\left(\frac{1}{0.112} \ell n(35/40)\right)$$

$$= F_U(-1.16) = 0.122$$

$$0.01 = P[F \le f] = F_U(-2.33) \quad \text{or}$$

$$-2.33 = \frac{1}{0.112} \ell n(f/40) \qquad f = 31 \text{ ksi}$$

3.26

$$\ln \check{m}_Y = \ln m_Y - \frac{1}{2}\sigma^2_{\ln Y}$$

$$\ln \check{m}_Y = 5.031 - 0.435 = 4.596$$

$$P[X \geq 400] = P[Y \geq 400 - 12.8] = 1 - F_Y(387.2)$$

$$= 1 - F_U\left(\frac{\ln(387.2)-4.596}{0.933}\right)$$

$$= 1 - F_U(1.46)$$

$$= 1 - 0.928$$

$$= 0.072$$

The two parameter distributions in 3.19 led to probability estimates of about 0.2.

3.27 $P[\text{annual max} \geq 15{,}000] = 1 - F_X(15{,}000)$

$$m_Y = u + \frac{0.577}{\alpha}$$

$$\sigma_Y = \frac{1.282}{\alpha} = 3000$$

$$\alpha = 0.000428$$

$$10{,}000 = u + \frac{0.577}{0.000428} = u + 1345$$

$$u = 8655$$

$$1 - F_X(15{,}000) = 1 - \exp[-e^{-0.000428(15{,}000-8655)}]$$

$$= 1 - \exp[-e^{-2.72}]$$

$$= 0.064$$

3.27 (continued)

CDF max over 20 years Y

$$F_Y(y) = P[Y \le y] = [F_X(y)]^{20} = P[\text{all 20 annual max} \le y]$$

$$= \{\exp[-e^{-0.000428(y-8655)}]\}^{20}$$

$$F_Y(y) = \exp[-20e^{-0.000428(y-8655)}]$$

Note (replace 20 by $e^{\ell n 20}$) the lifetime maximum also has a Type I distribution with the same α and with u increased by $(\ell n 20)/\alpha$.

$$1 - F_Y(15{,}000) = 1 - \exp[-20e^{-0.000428(15{,}000-8655)}]$$

$$= 1 - \exp[-20(0.0663)]$$

$$= 0.734$$

3.28 Type I: $1 - \frac{1}{m} = \exp[-e^{-\alpha(x_m-u)}]$

For large m

$$1 - \frac{1}{m} \cong 1 - e^{-\alpha(x_m-u)}$$

$$-\ell nm = -\alpha(x_m-u)$$

$$x_m = \frac{1}{\alpha}\ell nm + u$$

Type II:

$$1 - \frac{1}{m} = e^{-(\frac{u}{x_m})^k}$$

$$1 - \frac{1}{m} = 1 - (\frac{u}{x_m})^k$$

$$(\frac{1}{m})^{1/k} = \frac{u}{x_m}$$

$$x_m = um^{-1/k}$$

3.29 In contrast to continuous random variables, when the variables are discrete there is a finite probability that two or more of n variables will take on identical values. Thus

$$p_{Y_n}(y) = P[\text{maximum of } X_1, X_2, \ldots X_n = y]$$

$$= \sum_{k=1}^{n} P[\text{exactly k of variables}=y]P[\text{the remaining n-k variables} < y]$$

There are $\binom{n}{k}$ equally probably ways, different sets of k variables in the n.

$$p_{Y_n}(y) = \sum_{k=1}^{n} \binom{n}{k} p_X^k(y)[F_X^*(y)]^{n-k}$$

This is the binomial expansion of $[p_X(y) + F_X^*(y)]^n$ except it is lacking one term (k = 0); therefore

$$p_{Y_n}(y) = [p_X(y) + F_X^*(y)]^n - [F_X^*(y)]^n$$

But $p_X(y) + F_X^*(y) = F_X(y)$. Thus

$$p_{Y_n}(y) = [F_X(y)]^n - [F_X^*(y)]^n$$

This last result could also be obtained directly by arguing that

$$p_{Y_n}(y) = P[\text{all n of the } X_i \leq y] - P[\text{all n of the } X_i < y]$$

$p_{Y_n}(y)$ does not equal $np_X(y)[F_X(y)]^{n-1}$ because the events such as $[(X_i=y) \cap (\text{all others} \leq y)]$ and $[(X_j=y) \cap (\text{all others} \leq y)]$ are not mutually exclusive and the probability of their intersection, i.e., $[(X_i=y) \cap (X_j=y) \cap (\text{all others} \leq y)]$ is not zero, as it is in the continuous case.

3.29 (continued)

Special case:

$F_X^*(10) = 0$

$F_X^*(15) = F_X(10) = 0.8$

$F_X^*(20) = F_X(15) = 0.95$

$F_X(20) = 1.00$

$p_{Y_{10}}(10) = 0.8^{10} = 0.108$

$p_{Y_{10}}(15) = 0.95^{10} - 0.8^{10} = 0.492$

$p_{Y_{10}}(20) = 1 - 0.95^{10} = 0.400$

3.30 $\sigma_y \simeq \frac{1.282}{\alpha} = 6.2$

$\alpha = 0.207$

$m_y \simeq u + \frac{0.577}{\alpha} = 2$

$15.6 \simeq u + \frac{0.577}{0.207}$

$u = 12.81$

$P[\text{extreme gust} \geq 60] = 1 - F_Y(60) = 1 - \exp[-e^{-\alpha(60-u)}]$

$= 1 - \exp[-e^{-0.207(60-12.81)}]$

$= 0.00006$

500 observations are insufficient to verify the model so far into the tail (Sect. 4.5), but lacking evidence to the contrary, it may very well be appropriate to adopt this "extrapolation" since there is at least some phenomenological and asymptotic theory (Sect. 3.3.3) to support the model. In practical engineering situations there may be no recourse to such extrapolations.

3.31 a) The exact (non-asymptotic) distribution of Y is (n = 20,000)

$$F_Y(y) = [F_D(y)]^n = [1 - e^{-(cd)^k}]^n$$

Owing to the exponential form of F_D, the asymptotic distribution must be of Type I i.e., of the form

$$F_Y(y) = e^{-e^{-\alpha(y-u)}}$$

A direct application of the method outlined in Appendix B will yield

$$\alpha = ck(\ell n\ n)^{1-1/k} = 10.6$$

$$u = \frac{1}{c}(\ell n\ n)^{1/k}$$

Alternatively, observe that for large n, $f_Y(y)$ will be distributed along the y axis primarily in a region in the upper tail of $f_D(y)$, i.e., where $F_D(y)$ is close to 1 or where $\exp(-(cy)^k)$ is small. This implies

$$1 - e^{-(cy)^k} \cong \exp(-e^{-(cy)^k})$$

or

$$F_Y(y) \cong \exp(-ne^{-(cy)^k}) = \exp(-e^{-cy^k + \ell n\ n})$$

To get the Type I form y^k must be replaced by a linear function. As in Appendix B, y^k can be expanded about the mode of y or u_n (that is, the value of y such that $1 - F_D(y) = 1/n$), i.e., about $\frac{1}{c}(\ell n\ n)^{1/k}$.

Expansion and collection of terms will yield the same results as above.

3.31 (continued)

b) The 50-year displacement is that annual maximum exceeded with probability 1/50.

$$\frac{1}{50} = 1 - F_Y(y_{50}) = 1 - \exp(-e^{-\alpha(y_{50}-u)})$$

$$\frac{1}{50} \cong e^{-\alpha(y_{50}-u)}$$

$$\alpha(y_{50}-u) = \ell n\ 50$$

$$y_{50} = u_n + \frac{1}{\alpha}\ \ell n\ 50$$

$$= 0.63 + \frac{1}{10.6}\ \ell n\ 50 = 1.00$$

Using the exact distribution $y_{50} = 1.03$, implying the Type I asymptotic distribution is a good approximation even as far out into the tail as the y_{50} value.

3.32

X_1, X_2, and X_3 have a multinomial distribution:

$$p_{X_1,X_2,X_3}(x_1,x_2,x_3) = \frac{8!}{X_1!X_2!X_3!}(0.4)^{x1}(0.1)^{x2}(0.5)^{x3}$$

$$\text{for } x_1+x_2+x_3 = 8$$

The probability half go west and half north is

$$p_{X_1,X_2,X_3}(4,0,4) = \frac{8!}{4!0!4!}(0.4)^4(0.1)^0(0.5)^4 = 0.112$$

$$p_{X_1,X_3}(x_1,x_3) = \sum_{\text{all } x_2} p_{X_1,X_2,X_3}(x_1,x_2,x_3)$$

$$= \sum_{x_2=0}^{8} \frac{8!}{x_1!x_2!x_3!}(0.4)^{x1}(0.1)^{x2}(0.5)^{x3}$$

$$= 1.105\ \frac{8!}{x_1!x_3!}(0.4)^{x1}(0.5)^{x3} \qquad \text{for } 0 \le x_1+x_3 \le 8$$

3.32 (continued)

The marginal distribution of X_1 is Binomial:

$$P[X_1 > 4] = \sum_{x=5}^{8} \binom{8}{x} (0.4)^x (0.6)^{8-x} \qquad B(8, 0.4)$$

$$= 0.173$$

3.33 Distribution of the size of largest of many flaws is Type I (largest values) extreme value:

$$F_X(x) = \exp[-e^{-\alpha(x-u)}] \qquad -\infty \leq x \leq \infty$$

$$S = s_o - cX$$

$$F_S(s) = P[S \leq s] = P[X \geq \frac{s_o - s}{c}]$$

$$= 1 - F_X(\frac{s_o - s}{c})$$

$$F_S(s) = 1 - \exp[-e^{-\alpha(\frac{s_o - s}{c} - u)}]$$

$$= 1 - \exp[-e^{\frac{\alpha}{c}(s - s + uc)}] \qquad -\infty \leq s \leq \infty$$

S has a Type I (smallest values) extreme value distribution (Eq. 3.3.58) with parameters α/c and $(s-uc)$.

3.34 Y is lognormal

and $\ell n\ Y = \ell n\ F - \ell n\ A$ is normal.

$$m_{\ell n\ Y} = m_{\ell n\ F} - m_{\ell n\ A}$$

$$= \ell n\ \breve{m}_F - \ell n\ \breve{m}_A$$

$$= \ell n(\frac{\breve{m}_F}{\breve{m}_A}) = \ell n\ \breve{m}_Y$$

$$\sigma^2_{\ell n\ Y} = \sigma^2_{\ell n\ F} + \sigma^2_{\ell n\ A}$$

3.35 Observe m interarrival times with exponential distribution

$$F_T(t) = 1 - e^{-\lambda t}$$

a) Let Y_m be smallest interarrival time

$$1 - F_{Y_m}(y_m) = P[\text{all } t \geq y_m] = [1 - F_T(y_m)]^m$$

$$= e^{-m\lambda y_m} \Rightarrow F_{Y_m}(y_m) = 1 - e^{-m\lambda y_m}$$

which is exponential with parameter $m\lambda$, i.e., the expected value of the minimum is $1/m$ times the mean interarrival time.

b) The sum of n independent variables with common exponential distribution is gamma. Therefore

$$m_Z = n/m\lambda = 5/m\lambda$$

$$\sigma_Z^2 = n/m^2\lambda^2 = 5/m^2\lambda^2$$

implying Z is $G(5, m\lambda)$

3.36 a)

$$\Pi = \begin{vmatrix} 0 & 2/3 & 1/3 & 0 \\ 2/3 & 0 & 0 & 1/3 \\ 1/3 & 0 & 0 & 2/3 \\ 0 & 1/3 & 2/3 & 0 \end{vmatrix}$$

b) The Markov assumption implies passengers' choices of destination are independent. If there were dependence of choices (e.g., if passengers came in pairs for the same city), the conditional probability of the next state would depend on (at least) the previous as well as the previous state. E.g., for paired passengers

$$P[X(t+1) = 4 | (X(t) = 2) \cap (X(t) = 1)] = 0$$

while

$$P[X(t+1) = 4 | (X(t) = 2) \cap (X(t) = 4)] = 1$$

3.36 (continued)

c) Any 3 of

$$q_1^* = 2/3q_2^* + 1/3q_3^*$$

$$q_2^* = 2/3q_1^* + 1/3q_4^*$$

$$q_3^* = 1/3q_1^* + 2/3q_4^*$$

$$q_4^* = 1/3q_2^* + 2/3q_3^*$$

and

$$q_1^* + q_2^* + q_3^* + q_4^* = 1$$

Solution: all $q_i^* = 1/4$.

3.37 Poisson arrivals with rate λ

$$p_{N(t)}(n) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}$$

a) Let V(t) = number of vehicles in garage at time t .
If the garage is not full, V(t) = X + N(t). If the garage is full, V(t) = m. The last event occurs if $N(t) \geq m - X$. Therefore

$$p_{V(t)}(v) = \begin{cases} \sum_{x=0}^{v} p_{N(t)}(v-x)p_X(x) = \sum_{x=0}^{v} \frac{(\lambda t)^{v-x} e^{-\lambda t}}{(v-x)!} \frac{q(1-q)^x}{1-(1-q)^{m+1}} \\ \qquad\qquad \text{for } 0 \leq v \leq m-1 \\ \sum_{x=0}^{m} P[N(t) \geq m-x]p_X(x) = \sum_{x=0}^{m} \frac{q(1-q)^x}{1-(1-q)^{m+1}} \sum_{n=m-x}^{\infty} \frac{(\lambda t)^n e^{-\lambda t}}{n!} \\ \qquad\qquad \text{for } v = m \end{cases}$$

3.37 (continued)

b) Using last equation:

$$p_{V(t)}(5) = \frac{(0.75)}{1-(0.25)^6}(1-F_X(4)) + \frac{(0.75)(0.25)}{1-(0.25)^6}(1-F_X(3)) + \ldots + \frac{(0.75)(0.25)^5}{1-(0.25)^6}$$

For $\lambda = 0.20/hr$; $p_{V(t)}(5) = 7 \times 10^{-4}$

For $\lambda = 2.0/hr$; $p_{V(t)}(5) = 4 \times 10^{-3}$

For $\lambda = 20/hr$; $p_{V(t)}(5) = 0.60$

Inspection of the individual terms reveals that for small λ, virtually the only chance that the garage will be full is if it starts that way; for intermediate λ, all terms, i.e., all combinations of initial state and number of arrivals contribute to the probability of being full; for larger λ, in most cases the garage will be full because it started empty and 5 or more cars arrived.

3.38 a) $p_{13}^{(2)} = p_{13}p_{33} + p_{12}p_{23} + p_{11}p_{13}$

$= 0.36$

b) Formally

$$q_1^* = 0.6q_1^* + 0.2q_2^*$$
$$q_2^* = 0.2q_1^* + 0.6q_2^*$$
$$q_3^* = 0.2q_1^* + 0.2q_2^* + q_3^*$$
$$q_1^* + q_2^* + q_3^* = 1$$

Solving formally or by inspection:

$$q_1^* = 0, \quad q_2^* = 0, \quad q_3^* = 1$$

3.38 (continued)

c) State 3 is a trapping state because $p_{33} = 1$. Examples of processes with such trapping states include various kinds of deteriorating systems with a final failure state.

3.39 $M = C - D$

$$m_M = m_C - m_D$$

$$\sigma_M^2 = \sigma_C^2 + \sigma_D^2$$

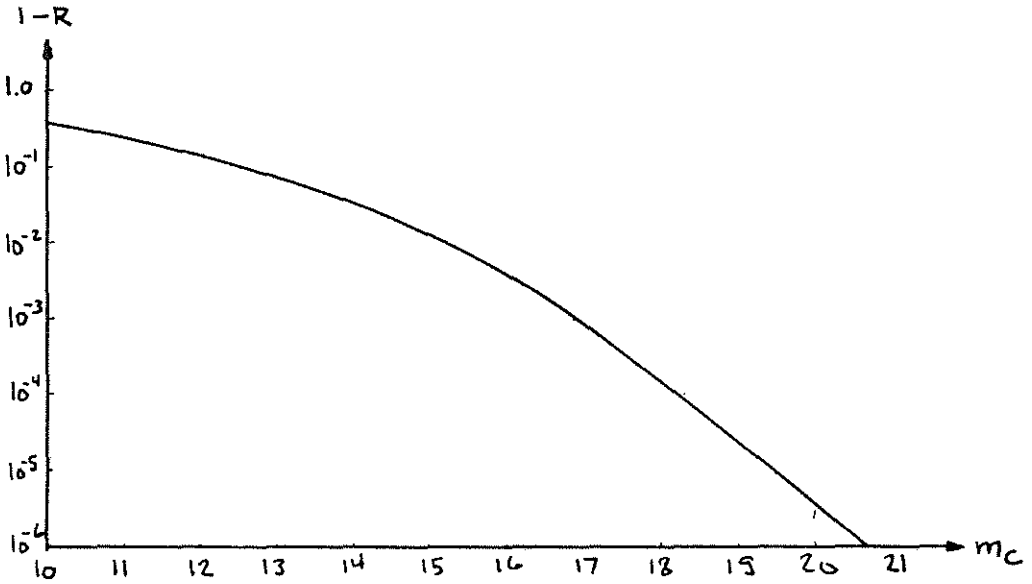
$$V_m = \frac{\sqrt{\sigma_C^2 + \sigma_D^2}}{m_C - m_D}$$

A linear combination of independent normal variables is normal.

$$R = P[D \leq C] = P[C-D \geq 0] = 1 - P[M \leq 0] = 1 - F_M(0)$$

$$= 1 - F_U(-\frac{m_M}{\sigma_M}) = 1 - F_U(-\frac{1}{V_m})$$

$$R = 1 - F_U(\frac{m_D - m_C}{\sqrt{\sigma_C^2 + \sigma_D^2}})$$



3.40 $F = \frac{C}{D} = CD^{-1}$ Lognormal

$R = P[F \geq 1] = 1 - F_F(1)$

$m_{\ell n\ F} = m_{\ell n\ C} - m_{\ell n\ D}$

$= \ell n\ (\frac{\breve{m}_C}{\breve{m}_D}) = \ell n\ \breve{m}_F = \ell n(200/100) = 0.694$

$\sigma^2_{\ell n\ F} = \sigma^2_{\ell n\ C} + \sigma^2_{\ell n\ D}$

$\sigma^2_{\ell n\ F} = \ell n\ (v_C^2+1) + \ell n(v_D^2+1)$

$= \ell n\ (1.01) + \ell n\ (1.0225)$

$= .01 + .0223 = 0.0323$

$\sigma_{\ell n\ F} = 0.1796$

$F_F(1) = F_U(\frac{\ell n\ 1 - \ell n\ \breve{m}_F}{\sigma_{\ell n\ F}})$

$= F_U(\frac{-0.694}{.1796}) = 0.00012$

$R = 1 - F_F(1) = 1 - 0.00012$

$R = 0.99988$

3.41 a) $A = Z^2$

Since $\ell n\ A = 2\ell n\ Z$, $\ell n\ A$ is normal, or A is lognormal with

$\sigma^2_{\ell n\ A} = 4\sigma^2_{\ell n\ z}$

$\breve{m}_A = (\breve{m}_Z)^2$

3.41 (continued)

b) $A = Z, Z_2$

Since $\ell n\ A = \ell n\ Z_1 + \ell n\ Z_2$, $\ell n\ A$ is normal and A is lognormal with

$$\breve{m}_A = (\breve{m}_Z)^2$$

$$\sigma^2_{\ell n\ A} = 2\sigma^2_{\ell n\ Z}$$

c) Model (a) has a larger dispersion.

3.42 a)

State 0: 0 cars in lane
State 1: 1 car in lane
State 2: 2 cars in lane

$$\Pi = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{vmatrix}$$

b) $q_0^* = \frac{1}{2}q_0^* + \frac{1}{2}q_1^* + \frac{1}{2}q_2^*$

$q_1^* = \frac{1}{2}q_0^* + \frac{1}{2}q_2^*$

$q_2^* = \frac{1}{2}q_1^*$

$q_0^* + q_1^* + q_2^* = 1$

Solving the last equation and any two of the first three equations:

$q_0^* = \frac{1}{2}$; $q_1^* = 1/3$; $q_2^* = 1/6$

The lane will be full an average of 1/6 of the times it is "observed."

3.43 Let X_i = no. recorded in i^{th} region in t years.

$$p_{X_i}(x) = (\lambda t)^x e^{-\lambda t}/x!$$

Solution A: $Y = \Sigma X_i$, the sum of Poisson variables, is also Poisson distributed with parameter $4\lambda t$. Therefore

$$p_Y(1) = (4\lambda t)e^{-4\lambda t}$$

Solution B: $P[(X_i=1)$ and the other $X_j=0] = p_X(1)[p_X(0)]^3$

$$= \lambda t e^{-\lambda t}(e^{-\lambda t})^3 = \lambda t e^{-4\lambda t}$$

but there are four mutually exclusive ways to pick which of the X_i's is to equal 1.

Solution C: Let Y_0 = no. of regions with 0 events, Y_1 = no. of regions with 1 event, and Y_2 = no. of regions with 2 or more events. The Y_i's have a multinomial distribution with $p_o = e^{-\lambda t}$, $p_1 = \lambda t e^{-\lambda t}$, and $p_2 = 1 - p_o - p_1$. The event of interest has a probability equal to $p_{Y_o,Y_1,Y_2}(3,1,0)$. The result is as above.

3.44 Soil C

$$P[r\ell_C s X_C \leq ew] = P[X_C \leq \frac{ew}{rs\ell_C}]$$

$$= P[\frac{x_C - m_C}{\sigma_C} \leq \frac{ew/rs\ell_C - 1.5ew/rs\ell_C}{0.20x1.5ew/rs\ell_C}]$$

$$= F_U(\frac{1-1.5}{0.3}) = F_U(-1.66)$$

$$= 0.04846$$

3.44 (continued)

Soil A+B

$$P[r(X_A \ell_A s + X_B \ell_B s) \leq ew]$$

$$P[0.3X_A + 0.7X_B \leq \frac{ew}{rs\ell_C}]$$

Let $0.3X_A + 0.7X_B = Z$

$$m_Z = 0.3m + 0.7m = m$$

$$\sigma_Z^2 = 0.09\sigma_{X_A}^2 + 0.49\sigma_{X_B}^2 + 2(0.5)(0.3)(0.7)\sigma_{X_A}\sigma_{X_B}$$

$$= v^2m^2[0.09 + 0.49 + 0.21] = 0.79v^2m^2$$

$$\sigma_Z = 0.89vm = (0.89)(0.20)1.5\,\frac{ew}{rs\ell_C} = 0.268\,\frac{ew}{rs\ell_C}$$

$$P[Z < \frac{ew}{rs\ell_C}] = P[\frac{Z-m_Z}{\sigma_Z} < \frac{ew/rs\ell_C - 1.5ew/rs\ell_C}{0.268ew/rs\ell_C}$$

$$= F_U(\frac{1-1.5}{0.268}) = F_U(-1.86)$$

$$= 0.03144$$

Embankment A+B is safer than soil C, but they have equal mean resistances. The relative safety of A+B vanishes as the correlation coefficient approaches 1. This coefficient is a measure of spatial homogeneity of the soil (see Prob. 3.23).

3.45

a) No. $P[Z(n+1) = j \mid (Z(n) = i) \cap (Z(n-1) = k) \cap \ldots]$

depends on the values of Z(n-1), Z(n-2), ... A little thought will justify this. Note that the values of Z(n) do not always tell one uniquely how much water is left in the dam. A release of 2 may be associated with a dam left empty or full. Thus knowledge of past releases will alter the likelihoods of future releases. For example,

$$P[Z(n+1) = 2 \mid (Z(n) = 2) \cap (Z(n-1) = 3)] \neq$$

$$P[Z(n+1) = 2 \mid (Z(n) = 2) \cap (Z(n-1) = 0)]$$

In the first case, knowledge that Z(n-1) = 3 implies the dam was left full, while knowledge that Z(n-1) = 0 implies the dam was left empty. Knowing in addition that Z(n) = 2, in the former case implies a low probability (0.1) that the dam is now (n) empty (it could happen only if the inflow Y_n was exactly 0); in the latter case Z(n-1) = 0 and Z(n) = 2 imply that the dam is now empty with high probability ($P[Y_n = 2]/P[Y \geq 2] = 0.3/0.6 = 0.5$; the conditional probability of an inflow of exactly 2 given the 2 or more inflow units implied by an empty dam followed by a release of 2). The probability of future release (Z(n+1)) of 2 will, of course, be lower if the probability of an empty dam is high.

b) Taking the long run probabilities of the dam states from page 345:

$$P[2 \text{ or more released}] = 1 - P[0 \text{ released}] - P[1 \text{ released}]$$

$$P[0] = P[\text{dam empty} \cap \text{no inflow}]$$

$$= (0.25)(0.1) = 0.025$$

$$P[1] = P[(\text{dam empty}) \cap (1 \text{ unit inflow})] + P[(\text{dam has 1 unit}) \cap (\text{no inflow})]$$

$$= (0.25)(0.3) + (0.25)(0.1)$$

$$= 0.1$$

3.45 (continued)

$$P[\text{2 or more released}] = 1 - 0.025 - 0.1$$

$$= 0.875$$

Without dam: P[release would be insufficient in any year] = 0.40
With dam: P[release would be insufficient in any year] = 0.125

c) Long-run, expected annual costs:

Without dam: (1,000,000)(0.1) + (100,000)(0.3) = 130,000
With dam: (100,000) + (1,000,000)(0.025) + (100,000)(0.1)
= 135,000

Decision: Dam should not be built.

3.46

a) The result is incorrectly stated in the first printing. It should read

$$F_{Y(i)}(y) = P[\text{at least } i \text{ of the } X\text{'s are less than or equal to } y]$$

$$= \sum_{j=i}^{n} P[\text{exactly } j \text{ of the } X\text{'s are less than or equal to } y]$$

$$= \sum_{j=i}^{n} \binom{n}{j} [F_X(y)]^j [1 - F_X(y)]^{n-j}$$

The verbal interpretation is given by stating the second line in words.

b) Let C = system capacity

$$F_C(c) = F_{Y(j)}(c) = \sum_{k=j}^{n} \binom{n}{k} [F_X(c)]^k [1 - F_X(y)]^{n-k}$$

3.46 (continued)

c) Let T = time between replacements = $Y^{(j)}$

$$F_T(t) = F_{Y^{(j)}}(t) = \sum_{k=j}^{n} \binom{n}{k} (1-e^{-\lambda t})^k e^{-\lambda t(n-k)}$$

Note that owing to the memoryless property of the exponential distribution (page 244), the conditional distribution of the remaining lives of the n-j lights still burning at the replacement is again exponential. Thus they could be left in without reducing the effectiveness of the system. In practice, of course, although often used in such maintenance studies, the exponential distribution cannot explain time-to-failure when wear-out failures are involved. See Prob. 2.69.

3.47 a) $$p_X(x) = \int_{n=0}^{\infty} p_{X|N}(x,n)\, f_N(n)\, dn$$

$$= \int_{n=0}^{\infty} \frac{n^x e^{-n}}{x!} \frac{\mu(\mu n)^{k-1} e^{-\mu n}}{(k-1)!}\, dn$$

Integrating and re-arranging

$$= \frac{\Gamma(k+x)}{x!\Gamma(k)} \left(\frac{\mu}{\mu+1}\right)^k \left(\frac{1}{\mu+1}\right)^x \qquad x = 0, 1, 2, \ldots$$

For integer k this is the form of the negative binomial appearing in the footnote on page 231. It is more generally defined for arbitrary (positive) k, just as its analog in Section 3.2, the gamma distribution, is defined for arbitrary k, gamma functions replacing factorials.

3.47 (continued)

b) From Eq. 3.1.13 and 3.1.4 (recalling that these equations are for a variable equal to X+k)

$$m_X = \frac{k}{p} - k = k\left(\frac{\mu+1}{\mu}\right) - k = k/\mu$$

$$\sigma_X^2 = k(1-p)/p^2 = k\left(\frac{1}{\mu+1}\right)\left(\frac{\mu+1}{\mu}\right)^2 = k(\mu+1)/\mu^2$$

From the data

$$m_X \cong \bar{x} = 0.483 = k/\mu$$

$$\sigma_X^2 \cong s^2 = 1 = k(\mu+1)/\mu^2$$

Solving for the parameter estimates

$$\mu = 0.935$$

$$k = 0.452$$

Note that in this case the estimate of k is in fact non-integer. We observe that workers (mean) accident rates ν or N vary but an "average worker" has (mean) rate of $m_N = k/\mu = 0.483$ accidents per 3 months. This happens to also equal m_X.

Comparison of observed and predicted frequencies of workers with x accidents in 3 months:

x	Observed Frequency	Calculated Probability, $p_X(x)$
0	0.715	0.720
1	0.178	0.166
2	0.063	0.063
3	0.019	0.027
4	0.010	0.012
5	0.010	0.005
≥ 6	0.005	0.005

The fit is extremely good. Chapter 4 will formalize such comparisons.

3.48 a) The load-strain law of the member must be elastic (linear or non-linear). For an elastic system, the strain is a one-to-one function of the load; knowing the load one knows the strain, independently of past loading history. If the material is non-elastic, its past load-strain history influences how it will respond to the next load change, even if the present strain is known. For example, a permanent set may have been induced by previous loadings. Such a memory in the material is inconsistent with the Markov assumption; knowledge of past states will alter the conditional probabilities of future states.

b) Since there is a one-to-one correspondence between the load and the strain, the load process must be a Markov process if the deformation process is to be Markov. In particular, the successive loads need not be independent random variables.

3.49 a)

$$q_1^* = 0.4q_1^* + 0.3q_2^* + 0.1q_3^* + 0.1q_4^*$$

$$q_2^* = 0.3q_1^* + 0.3q_2^* + 0.4q_3^* + 0.4q_4^*$$

$$q_3^* = 0.2q_1^* + 0.2q_2^* + 0.3q_3^* + 0.4q_4^*$$

$$q_4^* = 0.1q_1^* + 0.2q_2^* + 0.2q_3^* + 0.1q_4^*$$

$$q_1^* + q_2^* + q_3^* + q_4^* = 1$$

Solve the last plus any three of the first four equations.

$$q_4^* = 0.16$$

$$q_3^* = 0.262$$

$$q_2^* = 0.345$$

$$q_1^* = 0.234$$

3.49 (continued)

b) The expected damage in any day is

$$[(0.16)(5\text{x}10^{-7}) + (0.262)(5\text{x}10^{-6}) + (0.345)(5\text{x}10^{-5}) + (0.234)(5\text{x}10^{-4})] = 1.36 \text{ x } 10^{-4}$$

In 30 years, there are 10,950 days; the expected total damage is the sum of the expected damages:

$$(1.36\text{x}10^{-4})(1.095\text{x}10^{4}) = 1.49$$

In 60 years, there are 21,900 days; the expected total damage is 2.98.

4.1 The estimator of the mean is $\overline{X}$.

$E[\overline{X}] = m_X = 50$

For n = 10

$$\text{Var}[\overline{X}] = \sigma_X^2/10 = \left(\frac{100^2}{12}\right)/10 = 83.3$$

4.2 a) The estimator is $\hat{P}$ = total failures/total no. of tests.
Here the observed estimate is $\hat{p} = 60/1500 = 0.04$.

b) $\text{Var}[\hat{P}] = \text{Var}[\text{No. of failures in } 1500]/1500^2 = 1500p(1-p)/1500^2$

An estimate of this variance is thus
$(0.04)(0.96)/1500 = 2.55 \times 10^{-5}$

Assuming $\hat{P}$ is approximately normally distributed, the 95% confidence interval is

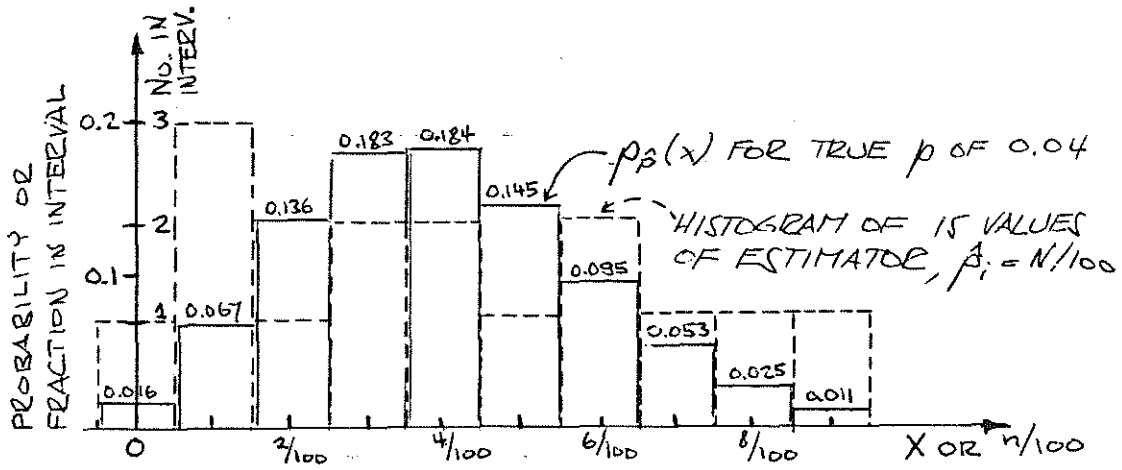
$$0.04 \pm 1.96\sqrt{2.55 \times 10^{-5}} = 0.04 \pm 0.01$$

c) N_i is binomial, B(100, p). Therefore $\hat{P} = N_i/100$ has a "binomial-like" distribution for non-integer values of the argument.

$$p_{\hat{P}}(x) = P[\hat{P} = x] = \binom{100}{100x} p^{100x}(1-p)^{100-100x} \quad \text{for } x = 0,\ 1/100,\ 2/100,\ \ldots,\ 100/100.$$

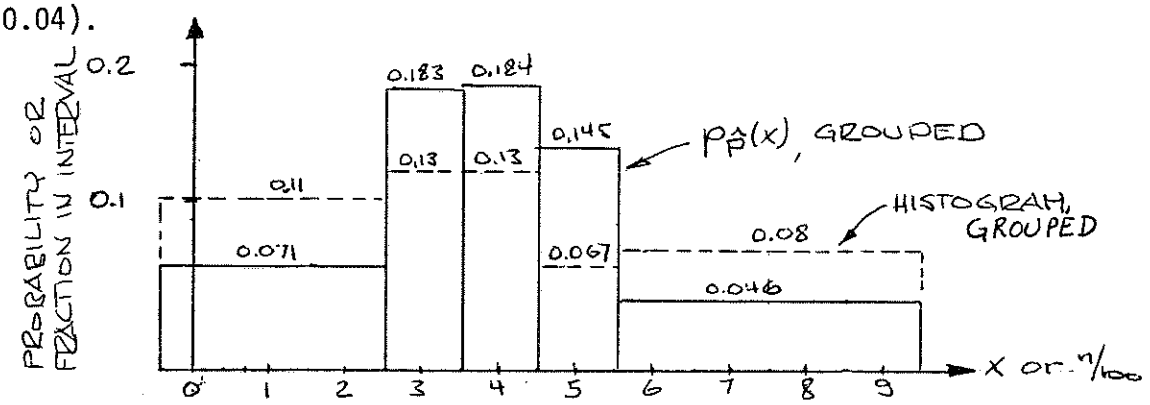
The variance of $\hat{P}$ (for n = 100) is p(1-p)/100 (see part b)

d)



4.2 (continued)

For small sample size (15), it is better to look at grouped data (here grouped into approximately equally likely intervals, given p = 0.04).



The sample variance of the 15 observations of the estimate of p is

$$s^2_{\hat{p}} = \frac{1}{15} \sum_{i=1}^{15} (\hat{p}_i - 0.04) = 0.00074$$

The variance of $\hat{P}$ for true p = 0.04 is (0.04)(0.96)/100 = 0.00038. Both the histogram and the sample variance show broader scatter in the estimator than sampling theory (statistics) predicts. The cause is probably systematic changes in p in certain lots due to changes in the production process. It is just such changes that the quality control program from which this data was taken was trying to catch. The true p when the production process is functioning properly might be only 0.02 to 0.03. Some or all of the lots with observed fractions of 0.06 or greater may in fact be from lots where the process was "out-of-control." See Problem 4.6.

4.3 a) $\hat{m} = \bar{x} = 7970/31 = 257$ cfs

$\hat{\sigma}^2 = s^2 = 1{,}120{,}000/30 = 37{,}300\ (\text{cfs})^2$

b) $\bar{X}$ is normally distributed; k = 1.96. The confidence limits on m_X are $257 \pm 1.96(180)/\sqrt{31} = 257 \pm 63.5$ cfs

c) By method of moments (there are other methods leading to somewhat different numerical estimates)

$\hat{m}_{\ell nX} = 181.8/31 = 5.87$

$\hat{\sigma}^2_{\ell nX} = 21.837/30 = 0.705$

$\hat{\hat{m}}_X = e^{\hat{m}_{\ell nX}} = e^{5.87} = 355$

4.4 a) $\sigma^2_{\hat{P}_i} = \text{Var}[N_i]/n^2 = np_i(1-p_i)/n^2 = p_i(1-p_i)/n$

Estimating $\sigma^2_{\hat{P}_i}$ by replacing p_i by its estimate, when an observed value of $\hat{P}_i$, namely $\hat{p}_i$, is observed, the confidence interval is

$$\hat{p}_i \pm k_{\alpha/2}\sqrt{\hat{p}_i(1-\hat{p}_i)/n}$$

in which $k_{\alpha/2}$ is found in the normal table in the usual way.

b) $c_\alpha \leq \gamma p_i$ implies

$$k_{\alpha/2}\sqrt{p_i(1-p_i)/n} \leq \gamma p_i$$

Solving for n,

$$n \geq k^2_{\alpha/2}(1-p_i)/\gamma^2 p_i$$

Note for smaller values of p_i, i.e., for rare events, the smaller p_i the larger n must be.

4.4 (continued)

c) $k_{\alpha/2} = 1.64$; $p_i \cong 0.15$; $\gamma = 0.2$; Substituting

$n \geq 385$

4.5 a)

i) $k_{\alpha/2} = 1.96$

$\sigma k_{\alpha/2}/\sqrt{n} = (0.195)(1.96)/\sqrt{4217} = 0.0059$

Confidence limits: -0.02 ± 0.0059

ii) Using t-test

$t_{\alpha/2;n} = t_{0.025;4217} \cong t_{0.025;\infty} = k_{0.025} = 1.96$

Confidence limits: same as (i) owing to large n.

b) -0.02 ft. = -0.24 in.; 0.04 ft. = 0.48 in.

$$P[-0.24 \leq X \leq 0.48] = P\left[\frac{-0.24-0.02}{0.195} \leq U \leq \frac{0.48-0.02}{0.195}\right]$$

$$= F_U(2.56) - F_U(-1.13) = 0.994 - 0.129 = 0.865$$

P[X exceeds tolerance] = 1 - 0.865 = 0.135

Virtually all exceedances will be below -0.02 ft. rather than above 0.04 ft.

4.6 $3868 \pm (3)(318)/\sqrt{10} = \{3567, 4170\}$

Lot 3 exceeds upper control limit.

P[Normal variable deviates from m by more than 3σ] = 0.0027

If the process is always in control, 100 days represent 100 Bernoulli trials with p = 0.00135. N = no. of unnecessary excursions.

$P[N \geq 1] = 1 - p_N(0) \cong np = (100)(0.0027) = 0.27$

4.7 a) H_o: p = 0.1 versus H_1: $p \geq 0.1$

Accept H_o if R < c; assuming independence, R is binomial.

$$\alpha = P[R \geq c | p=0.1] = \sum_{r=c}^{5} \binom{5}{r}(0.1)^r(0.9)^{5-r}$$

For c = 0; $\alpha = 1$

For c = 1; $\alpha = 1 - (1-p)^5 = 1 - (0.9)^5 = 0.408$

For c = 2; $\alpha = 1 - [(1-p)^5 + 5p(1-p)^4] = 0.08$

Thus to get <u>at most</u> $\alpha = 0.10$, choose c = 2, i.e., change the controls only if 2 or more specimens are less than 6" thick. (Note for discrete statistics one cannot necessarily obtain exactly the desired α-level.)

b) H_o: p = 0.2 versus H_1: p = 0.1

Reject H_o if n - R > c or R < n - c; i.e., accept if $R \geq n - c$

$$\alpha = P[R < n-c | p = 0.2] = \sum_{r=o}^{n-c-1} \binom{n}{r}(0.2)^r(0.8)^{n-r}$$

$$\beta = P[R \geq n-c | p = 0.1] = \sum_{r=n-c}^{n} \binom{n}{r}(0.2)^r(0.8)^{n-r}$$

Find, by trial and error, n and c such that $\alpha \leq 0.05$ and $\beta \leq 0.2$. For economy find the smallest value of n possible. Assume approximate normality of R in this case, with $m_R = np$ and $\sigma_R^2 = np(1-p)$, and p equal 0.1 or 0.2. Putting the statements above in terms of standardized normal variables

$$\alpha = 0.05 = P\left[\frac{R-m_R}{\sigma_R} < \frac{n-c-(0.2n)}{\sqrt{n(0.2)(0.8)}}\right]$$

4.7(continued)

$$\beta = 0.2 = P\left[\frac{R-m_R}{\sigma_R} \geq \frac{n-c-0.1n}{\sqrt{n(0.1)(0.9)}}\right]$$

From normal tables for 0.05 and 0.2 probabilities (these approximations can be improved; see Prob. 3.18)

$-1.65 = (n-c-0.2n)/\sqrt{0.16n}$

$0.84 = (n-c-0.1n)/\sqrt{0.09n}$

Solving simultaneously for n and c

n = 83!

c = 69

In short, test 83 specimens, reject H_o (accept the work) only if 83-69 (or 14) or less are o.k. It is unlikely that such a large sample size would be considered economical for all but a very large project. If a smaller value of n must be used the inspector will have to compromise on his desired α, or β, or both.

4.9 For $\nu = 7-1-2 = 4$, Table A.2 gives a value of about $\chi^2 = 7.8$ for a probability of 10% of being exceeded. $4.506 \leq 7.8$. Accept H_o. The p-level is 0.35 from Table A.2 for $\chi^2 = 4.5$, $\nu = 4$.

4.10 a) The paper requires a transformation to "straighten out" the parabolic functions of the CDF.

b) The sum of the two identically, uniformly distributed variables has a triangular distribution on the interval -1 to +1. Therefore the average is triangular on -0.5 to +0.5.

c) Assume the true value is the average, 15.2. Plot the deviations from 15.2. Note that, unlike normal paper, this paper can't be used without an independent prior estimation of the parameters.

4.11 See text page 284. Compare with normal/log-normal paper, Fig. 4.4.15. Either put a log-scale on the x axis or plot logs of the observed values on Type I paper.

4.12 With so large a mean a normal approximation to the Poisson is appropriate, using $\nu = \bar{x} = 220$. Note that for the Poisson $m = \sigma^2 = \nu$. Therefore the data plotted on normal probability paper must be compared with a hypothesized line with $m = 220$ and $\sigma = \sqrt{220} = 14.8$. This σ is much smaller than the observed sample standard deviation of 74.2. This difference alone would probably cause rejection of the model. Physically, the clustering in time of small (magnitude 4 or less, say) aftershocks around large shocks causes larger variances than the Poisson process assumption permits. Screening of these correlated clusters of aftershocks (e.g., by considering only rarer, larger magnitudes) usually improves the comparison.

The K.S. statistic for $n = 29$ and $\alpha = 0.1$ is about 0.22. The model is rejected by the test.

4.13 Under the null hypothesis $p_X(x) = e^{-0.24}(0.24)^x/x!$

For $n = 29{,}531$:

x	$p_X(x)=p_i$	(obs. no.$-np_i$)	$(\text{obs. no.}-np_i)^2/np_i$
0	0.786	~~-661~~ 651	~~0.79~~ 18
1	0.189	-1067 −1072	207 206
2	0.0227	266 267	106 107
3	0.00181	107 106	216 218
4	0.00011	30 30	280 276
≥ 5	0.5×10^{-5}	18 18	2150 1983
			$\sum$ ~~2959~~ 2802

For $\alpha = 10\%$ and $\nu = 6\text{-}1\text{-}1 = 4$ (Table A.2) $\chi^2 = 7.8$. A clearcut rejection of the model is indicated. There are clearly many more multi-accident prone drivers than the model suggests. One conclusion is that fair insurance rates should <u>not</u> be independent of past driving performance.

4.16 $n = (40)(40)/80 = 20$; at $\alpha = 0.05$, critical value = 0.29. The observed value of the test statistic is apparently about 0.23 (in the region of $x = 350$ cfs). Therefore the model produced data which are not significantly different from the historic data.

4.18 $V_{\bar{X}_i} = \sigma_{\bar{X}_i}/m_{\bar{X}_i} = (\sigma_i/\sqrt{n_i})/m_i = V_i/\sqrt{n_i} = \text{constant} = c$

or $\sqrt{n_i} = V_i/c$ or $n_i = V_i^2/c^2$

But $n = \sum_{j=1}^{r} n_j = \frac{1}{c^2}\sum V_j^2 = \frac{n_i}{V_i^2}\sum V_j^2$

Solving for n_i gives the result sought.

Application gives, for example, $n_{\substack{\text{LISBON}\\\text{LOAM}}} = 100\,\frac{(.264)^2}{1.845} = 4.1$; use 4.

4.19 Under null hypothesis of independence

$P[A \cap B] = P[A]P[B] = p_1p_2$; etc.

The estimate of $P[A\ B] = \hat{p}_1\hat{p}_2 = (\frac{a+c}{n})(\frac{a+b}{n}) = n_3n_1/n^2$

The (estimate of the) expected number of observations of $A \cap B$ is $n\hat{p}_1\hat{p}_2 = n_3n_1/n$. The difference between the observed and expected number is

$a - n_3n_1/n$

The contribution to the χ^2 statistic of this event is

$(\text{observed-expected})^2/\text{expected} = (a-n_3n_1/n)^2/(n_3n_1/n)$

Doing the same for $A \cap B^c$, etc., and adding the four contributions yields c, as given in the text. The statistic has 4-1-2 = 1 degree of freedom since the 2 model parameters p_1 and p_2 are estimated from the data.

Application: $c = [(4)(0) - (5)(9)]^2 18/(13)(5)(9)(9) = 6.9$

$\chi^2_{0.05;1} = 3.84$ (Table A.2)

Reject the hypothesis, i.e., the data suggest that earlier failures "tend" to be compression failures.

4.20 a) $f_X(x) = 1/\theta \qquad 0 \le x \le \theta$

$L(\theta|x_1, \ldots x_n) = f_X(x_1)\ldots f_X(x_n) = 1/\theta^n \qquad x_j \le \theta \le \infty$

where x_j is the largest of the $x_1,\ldots x_n$. This problem is unfamiliar because the parameter appears in the limits on the regions over which $f_X(x)$ is defined.

4.20 (continued)

b) Maximize $L = 1/\theta^n$ over the interval $x_j \leq \theta \leq \infty$. This is a constrained maximization problem. Clearly the smaller θ the larger L. Therefore the maximum occurs at $\theta = x_j$, the sample maximum. This statistic is the maximum likelihood estimator of θ.

c) The mean of X is $\theta/2$; therefore the method of moments estimator of θ is $2\overline{X}$.

d) The sample maximum is 4. For a sample 1, 1, 2, 2, 4 the method of moments estimate is $2\overline{x} = 4$; for a sample of 1, 1, 1.5, 2, 4 this estimate is $2\overline{x} = 3.8$. Clearly the method of moments can lead to estimates of θ less than the largest observed value which is inconsistent with $f_X(x)$.

4.21 b) For such a small sample size the t-statistic should be used: $t_{.05;9} = 1.83$. Taking $\overline{x}$ and s* from the solution to Problem 1.1 (note that the values of s^2, not s^{*2}, are given in Chap. 1); apply Eq. 4.1.54.

Deflection: $0.1417 \pm \frac{1.83}{\sqrt{10}}\sqrt{(\frac{10}{9})0.000,209} = 0.1417 \pm 0.0088$ in.

Ultimate: 1847.5 ± 182 lb.

4.23 "Study the rule" implies at least evaluating the x_0 associated with each model, and calculating the probability that Oklahoma gets no water (i.e., $P[X \leq x_0]$). The conditional distribution of Oklahoma's water is of interest, too. The specific values found will depend on the parameter estimates used. These may come from one or more of various methods of estimation (e.g., method of moments or eyeball fit on probability paper).

4.24 Numerical results will depend on distribution adopted.

b) Sum products of probability of falling in each interval times percentage payment.

4.24 (continued)

c) Assuming no significant change in labor cost with pavement thickness, the contractor can reduce his expected costs (by reducing the mean--or"target"--pavement thickness) to the amount 0.4b (new mean/old mean), in which b is the price he bid to get the job. The expected payment will also decrease, however. This can be calculated by shifting the distribution chosen in (a) from its old to its new mean, and recalculating expected payment (= b times expected percentage). (Perhaps the variance should be reduced also--e.g., to maintain constant coefficient of variation.) At some point the contractor will find an optimal target thickness, significantly below 6 in.

4.26 a) See comments on solution of 4.21

Mean straight cables: $1.77 \pm (1.80)\sqrt{(12/11)(0.0559)}/\sqrt{12} = 1.77 \pm 0.13$

Mean curved cables: $2.25 \pm (1.75)\sqrt{(16/15)(0.0879)}/\sqrt{16} = 2.25 \pm 0.13$

For variances, use a two-sided form of Eq. 4.1.50:

With 90% confidence:

$$\frac{ns^2}{\chi^2_{.05,\ n-1}} \leq \sigma^2 \leq \frac{ns^2}{\chi^2_{.95,\ n-1}}$$

Straight cables:

$$\frac{12(0.0559)}{19.7} \leq \sigma^2 \leq \frac{12(0.0559)}{4.56} \quad \text{or } 0.034 \leq \sigma^2 \leq 0.144$$

Curved cables:

$$\frac{16(0.0879)}{25.0} \leq \sigma^2 \leq \frac{16(0.0879)}{7.25} \quad \text{or } 0.056 \leq \sigma^2 \leq 0.194$$

Notice that these two-sided limits on variance are symmetrical in terms of probability areas, not in the distance from the estimate of σ^2.

4.26 (continued)

c) Tests for equality of the means and variances (pg. 416) would be tests of the hypothesis that there are no differences between the two kinds of cables. If the hypothesis of equal means were rejected in favor of an alternate hypothesis that the mean of curved cables is larger, it would imply that curvature increases friction and hence force loss between jack and anchor. Similarly a larger curved-cable variance would imply that curvature leads to an additional source of variability. (All samples are from beams of similar length and configuration.)

4.28 a) If the loads on different floors are independent, the mean and variance of the total load on n columns is nm_1 and $n\sigma_1^2$ (in which m_1 and σ_1^2 are the mean and variance of the load on one floor). (See Problem 2.67.) The total area on n floors is na_1, where a_1 is the area of one floor (400 sq. ft.). Thus the mean and variance of Y, the load per unit area, are $nm_1/na_1 = m_1/a_1$ and $n\sigma_1^2/n^2a_1^2 = \sigma_1^2/na_1^2$. The former is independent of n, the latter decreases with n.

The simplest linear regression model (Section 4.3.2) (here with n the independent and Y the dependent variable, i.e., $E[Y]_n = \alpha + \beta n$) assumes that $\sigma^2_{Y_i}$ is independent of n. This is not the case in the independent loading model. Statistical analysis of the simplest regression model assumes independence of the Y_i, but with this data the loads associated with a particular bay are correlated; the total load on the eighth and ninth floors (n=2) is just the load on the ninth floor (n=1) plus a random increment. Therefore these two observations of Y (one given n=1, one given n=2) would be correlated.

The model described for total load or unit load is Markov with state load and "time" parameter n. (Compare with the running sum of heads model on text page 322.)

4.29 b) No; P[each gets 5000] = P[total $\leq$ 10,000]

c) P[user 1 gets his 5000] = P[total $\leq$ 5,000]
P[user 2 gets his 5000] = P[total $\leq$ 10,000]

d) No, water available this year is correlated with water available last year (see, for example, text pg. 342).

4.32 a) It is unlikely to yield samples representative of all the different density values within the layer. All the specimens are effectively measuring the same density value, hence their average value is little better an estimate of the true mean than a single specimen.

b) Owing to spatial correlation among closely neighboring soil volumes, the specimen densities are positively correlated. Presumably the closer they are together the more highly they are correlated. Therefore the variance of the estimator $\overline{X}$ (Eq. 4.1.12) is higher for the single hole method.

c) First method: $\text{Var}[\overline{X}] = \sigma^2/5$

Second method: $\text{Var}[\overline{X}] = \sigma^2/5 + \frac{2\sigma^2}{25}(0.9)(10)$

$$= \sigma^2 \frac{23}{25} = \sigma^2/1.1$$

The second method has the same variance as only 1.1 independent samples.

4.33 $\alpha = P[X > \text{minimum time} + c|H_o]$

$$= \frac{1}{2} e^{-c/3} + \frac{1}{3} e^{-c/5} + \frac{1}{6} e^{-c/8} = 0.10$$

Solving for c gives c = 10.5 minutes.

4.34 H_o: m = 4,550,000
H_1: m $\geq$ 4,550,000

$$= P[\overline{X} \geq c|m=4,550,000] = 1 - F_U\left(\frac{c-4,550,000}{\sigma_{\overline{X}}}\right)$$

4.34 (continued)

For the large size $\bar{X}$ is approximately normal even though σ is unknown (page 393); we replace $\sigma_{\bar{X}}$ by its estimate $s_X/\sqrt{n}$. Using $\alpha = 0.1$, $k_{0.1} = 1.28$

$$c = 4{,}550{,}000 \pm 1.28\sqrt{365{,}731 \times 10^6/52} = 4{,}657{,}000$$

The observed sample mean was 4,577,000, which is not significantly different from 4,550,000.

4.35 Calculated deflection $= \frac{P\ell^3}{48EI} = \frac{(394)(48)^3}{48(1.6\times10^6)(5.8)} = 0.098$ in.

H_o: $m_x = 0.098$ in.
H_1: $m_x \neq 0.098$ in.

$$\alpha = P[|\bar{X}-0.098| \geq c \,|\, m_X = 0.098] = P\left[|T| \geq \frac{c}{s/\sqrt{n}}\right]$$

in which T is t distributed with $\nu = n-1 = 9$. For $\alpha = 10\%$,

$$c = t_{0.05,9}\ s^*/\sqrt{n} = 1.83(0.0153)/\sqrt{10} = 0.0089$$

The observed difference $\bar{x} - 0.098 = 0.1417 - 0.098 = 0.044$.
The sample mean is significantly different from 0.098.

4.36 Following a two-sided version of page 416 example(for $\alpha = 0.1$)

H_o: $m_x = m_y$
H_1: $m_x \neq m_y$

$$P[|\bar{X} - \bar{Y}| \leq c \,|\, m_x = m_y] = 1 - \alpha$$

$$c = 1.64(2157 \text{ acre ft.})\sqrt{\frac{1}{8} + \frac{1}{9}} = 1730$$

The observed difference in the sample means is 11,500-14,700 = 3200, which implies a significant difference in the means.

4.37 a) $$L(\nu|x_1, x_2, \ldots x_n) = \prod_{i=1}^{n} p_X(x_i) = \prod e^{-\nu}\nu^{x_i}/x_i!$$

$$= e^{-\nu n}\nu^{\Sigma x_i}/\Pi x_i!$$

Removing factors independent of ν:

$$L(\nu) = e^{-\nu n}\nu^{\Sigma x_i}$$

4.37 (continued)

b) Find ν to maximize $L(\nu)$ or $\ell n\ L(\nu)$:

$$\ell n\ L(\nu) = -\nu n + \Sigma x_i\ \ell n\ \nu$$

$$\frac{d\ \ell n\ L(\nu)}{d\nu} = -n + (\Sigma x_i)/\nu = 0$$

$$\hat{\nu} = \Sigma x_i/n = \bar{x}$$

This is the same estimator as the method of moments suggests.

4.38 a) H_o: $m = 20$
H_1: $m = 30$
Statistic: $\bar{X}$
Accept H_o if $\bar{X} \leq c$.

b) $\alpha = 0.25 = P[\bar{X} \geq c|m=20] = 1 - F_U((c-20)\sqrt{5}/5)$
Assuming $\bar{X}$ is normal

$$c = 20 + k_{0.25}\ 5/\sqrt{5} = 20 + 0.67(5/\sqrt{5}) = 21.5$$

c) $\beta = P[\bar{X} \leq c|m=30] = P[\bar{X} \leq 21.5|m=30] = F_U((21.5-30)\sqrt{5}/5)$

$$= F_U(-3.80) \cong 0.$$

This suggests α could be reduced, unless the consequences of a type II error are much more serious than those of a type I error.

d) $\bar{x} = 27.6$; report that $m=30$.

4.39 States 0: empty
1: partially full
2: full

$\hat{p}_{00} = 15/35$; $\hat{p}_{01} = 15/35$; $\hat{p}_{02} = 5/35$
$\hat{p}_{10} = 20/45$; $\hat{p}_{11} = 20/45$; $\hat{p}_{12} = 5/45$
$\hat{p}_{20} = 0/20$; $\hat{p}_{21} = 10/20$; $\hat{p}_{22} = 10/20$

4.39 (continued)

In any case the nine estimators are negatively correlated since, for example, $\hat{p}_{00} + \hat{p}_{01} + \hat{p}_{02}$ must equal 1 (implying a too-high estimate of p_{00} will probably accompany a too-low estimate of p_{02}). Taking the observations consecutively adds further constraints, e.g., the number of observed transitions out of state 1 must equal (I1) the number of observed transitions into state 1. Negative correlation reduces the variance of estimators (Eq. 4.1.12).

4.40 $\tilde{X} = (X_1 X_2 \ldots X_n)^{1/n}$

$$\ell n\ \tilde{X} = \frac{1}{n} \Sigma \ell n\ X_i = \bar{Y}$$

where $Y_i = \ell n\ X_i$.

Since X_i is lognormally distributed, Y_i is normally distributed, and so is $\bar{Y}$ or $\ell n\ \tilde{X}$. Therefore $\tilde{X}$ is lognormal with

$$\breve{m}_{\tilde{X}} = (\breve{m}_X \breve{m}_X \ldots \breve{m}_X)^{1/n} = \breve{m}_X$$

$$\sigma^2_{\ell n\ \tilde{X}} = \sigma^2_{\ell n\ X}/n$$

Therefore (Eq. 3.3.31)

$$E[\tilde{X}] = \breve{m}_X \exp(\sigma^2_{\ell n\ X}/2n) \neq \breve{m}_X$$

Since $E[\tilde{X}] \neq \breve{m}_X$, $\tilde{X}$ is a biased estimator of $\breve{m}_X$, but as n grows $\exp(\sigma^2_{\ell n\ X}/2n)$ approaches 1. The arbitrariness of the commonly sought for unbiased estimator is again apparent. Although biased, X does have the property that its <u>median</u> is equal to the parameter it is estimating.

4.41 a) $E[\overline{X}] = m_X$; $\text{Var}[\overline{X}] = \sigma_X^2/n = m_X^2 V_X^2/n$

$V_{\overline{X}} = V_X/\sqrt{n}$

b) $\overline{X}$ is approximately normal:

$$1 - \alpha = P[m_X - k_{\alpha/2}\sigma_{\overline{X}} \le \overline{X} \le m_X + k_{\alpha/2}\sigma_{\overline{X}}]$$

$$= P[m_X(1-k_{\alpha/2}V_{\overline{X}}) \le \overline{X} \le m_X(1+k_{\alpha/2}V_{\overline{X}})]$$

$$= P[\overline{X}/(1+k_{\alpha/2}V_{\overline{X}}) \le m_X \le \overline{X}/(1-k_{\alpha/2}V_{\overline{X}})]$$

Confidence limits are

$$\{\overline{x}/(1+k_{\alpha/2}V_X/\sqrt{n});\ \overline{x}/(1-k_{\alpha/2}V_X/\sqrt{n})\}$$

Here $k_{\alpha/2} = 1.63$, $V_X = 0.2$, and $\sqrt{n} = 4$:

$$\{0.92\ \overline{x};\ 1.08\ \overline{x}\}$$

c) {93.5; 109.5}

4.42 a) H_o: $m = 10$

H_1: $m \neq 10$

$$1 - \alpha = P[|\overline{X}-m| \le c \,|\, m=10] = 0.90$$

$$c = k_{0.05}\sigma_{\overline{X}} = (1.65)(2)/\sqrt{4} = 1.65(1)$$

Observed mean $\overline{x} = 8$ lies outside 10 ± 1.65. Reject H_o.

b) The variance of $\overline{X}$ becomes (Eq. 4.1.12)

$$\sigma_{\overline{X}}^2 = \sigma_X^2/n + \rho(2)(\sigma_X^2)(6)/n^2 = (4 + 6)\sigma^2/16 = 2.5$$

$$\sigma_{\overline{X}} = \sqrt{2.5} = 1.48$$

$$c = (1.65)(1.48) = 2.45$$

$\overline{x} = 8$ lies within 10 ± 2.45. Accept H_o.

4.42 (continued)

c) H_o: $m \geq 10$ Reject if $\bar{X} \leq m-c$

H_1: $m < 10$

$$1 - \alpha = P[\bar{X} - m \geq -c \,|\, m=10]$$

$$c = k_{0.1}\sigma_{\bar{X}} = 1.28(1)$$

Observed mean lies below 10 - 1.28 = 8.72. Reject H_o.
(Under the assumptions of part b, the change to a one-sided test would cause the hypothesis to be rejected rather than accepted.)

4.43 a) Let Y = cost and r = no. of rooms

$$\widehat{E[Y]}_r = \hat{\alpha} + \hat{\beta}r = \$43,980 + \$8,560\ r$$

$\hat{\sigma} = \$82,470$ sample correlation coefficient $r_{Y,r} = 0.786$

$$\hat{\sigma}_{\bar{Y}_r} = \frac{\hat{\sigma}}{\sqrt{n}}\left[1 + \frac{(r-\bar{r})^2}{s_r^2}\right]^{\frac{1}{2}} = \frac{82,470}{\sqrt{11}}\left[1 + \frac{(r-33.4)^2}{135}\right]^{\frac{1}{2}}$$

e.g. at $r = \bar{r} = 33.4$ $\hat{\sigma}_{\bar{Y}_r} = \frac{82,470}{\sqrt{11}} = \$25,000$

at $r = 50$ $\hat{\sigma}_{\bar{Y}_r} = 25,000\left[1 + \frac{(50-33.4)^2}{135}\right]$

$= \$43,700$

b) For r = 32, an estimate of the expected cost is

$$\widehat{E[Y]}_r = 43,980 + (8560)(32) = \$319,000$$

An estimate of σ is \$82,470. But there is uncertainty in both these parameter estimates. To allow for this statistical uncertainty (as well as for the probabilistic uncertainty, i.e., σ) the conventional approach (pages 435 and 436)(in contrast to the Bayesian approach of Chap. 6) is to state 90%, say, confidence intervals on the cost of this particular job as

$$\widehat{E[Y]}_r \pm t_{\alpha/2;n-2}\sqrt{\sigma^2_{\bar{Y}_r} + \sigma^2} = 319,000 \pm 1.83(82,470)\sqrt{\frac{1}{11}\left[1 + \frac{(32-33.4)^2}{135}\right] + 1}$$

4.43 (continued)

b) $= 319{,}000 \pm 1.83(82{,}470)(1.05)$

$= 319{,}000 \pm 159{,}000$

The influence of parameter uncertainty on the limits is only 5% in this case, despite the small sample size.

c) $\widehat{E[Y]} = \hat{\alpha} + \hat{\beta}f = -\$124{,}250 + 121{,}750\ f$ in which f = no. of floors.

$\hat{\sigma} = \$87{,}440$ $\qquad r_{Y,f} = 0.756$

Even without consideration of parameter uncertainty it is clear that no. of floors will lead to somewhat wider 90% confidence bounds on the cost of this project. Multiple regression analysis permits the use of both number of floors and number of rooms to estimate cost. In this case the estimated regression line has been found to be

$$\widehat{E[Y]} = -\$98{,}500 + \$5{,}505\ r + \$65{,}561\ f$$

with $\hat{\sigma} = \$76{,}870$. In this case the use of both variables does not provide much improvement of the use of just one, rooms. The reason is that rooms and floors are also positively correlated, $r_{r,f} = 0.690$.

4.44 a) Let Y = COD and x = nitrogen content

$\widehat{E[Y]}_x = 370 + 35.2x$ $\qquad r_{x,y} = 0.76$

$\hat{\sigma} = 351$

b) $s_B^2 = s^2/ns_x^2 = 102$

H_o: $\beta = 0$

H_1: $\beta \neq 0$

Accept if $|B| \leq c$

$c = s_B t_{\alpha/2;n-2} = \sqrt{102}\ 2.26 = 22.8$

Observed value of B is $\hat{\beta} = 35.2$; conclusion is that the slope is not zero.

4.44 (continued)

c) $s_A^2 = \frac{s^2}{n}\left(1 + \frac{\bar{x}^2}{s_x^2}\right) = 246^2$

$H_o: \alpha = 0 \quad H_1: \alpha \neq 0$

Accept if $|A| \leq c$

$c = s_A t_{\alpha/2;n-2} = (246)(2.26) = 555$

Accept hypothesis since $\hat{\alpha}$ is $370 \leq 555$.

d) $\sigma^2 \leq \frac{ns^2}{\chi^2_{1-\alpha,n-2}} = \frac{(11)(351)^2}{4.2} = 2.62(351)^2$

$\sigma \leq (1.62)351$; $\sigma \leq 565$ with 90% confidence

e) The normal assumption appears reasonable.

f) The variance may in fact grow with x. One possible test is to make an F-test (page 416) for equality of the variance of the residuals for larger values of x and the variance for smaller values of x. There are theoretical difficulties, however, in the lack of independence of the residuals. See any intermediate text on applied regression analysis.

4.45 a) Strength = $e^{\alpha} z^{\beta}$

Let Y = ℓn(strength) and x = ℓn z

Y = ℓn(strength) = $\alpha + \beta$ ℓn z = $\alpha + \beta x$

Lightweight:

$\hat{\alpha} = 7.83$; $\hat{\beta} = 0.173$ $r_{x,y} = 0.95$

$\hat{\sigma} = 0.119$ This is, recall, the standard deviation of the natural logarithms, not the strength itself. Also this is the estimate of the two-specimen averages, not of individual specimens. Therefore one might estimate the σ of individual specimens as $\sqrt{2}\hat{\sigma} = 0.168$. Test for slope significance: $H_o: \beta \geq 1$. $H_1: \beta < 1$; Accept if $B \geq 1-c$

4.45 (continued)

$$c = s_B t_{\alpha,n-2} = \sqrt{s^2/ns_X^2}\, t_{0.1,6} = (0.119/\sqrt{8}(1.95))1.44 = 0.031$$

The observed value of $\hat{\beta}$, 0.173, is less than 1 - 0.031, therefore it is significantly less than 1. Note if the adjusted (doubled) $s^2(\hat{\sigma^2})$ is used, n should be doubled also, and t should be based, perhaps, on 12 or 14 degrees of freedom. There is little influence on c and none on the decision.

Confidence Limits

α: $7.83 \pm s_A t_{\alpha/2,n-2} = 7.83 \pm 0.099(3.0) = 7.83 \pm 0.3$

β: $0.173 \pm s_B t_{\alpha/2,n-2} = 0.173 \pm (0.023)(3.0) = 0.173 \pm 0.069$

The "adjusted" n will decrease the limits slightly through its influence on the degrees of freedom of t.

Ordinary Weight:

$\hat{\alpha} = 7.71$; $\hat{\beta} = 0.162$; $\hat{\sigma} = 0.162$

$s_B = 0.031$; $s_A = 0.135$; $r_{x,y} = 0.90$

Test for Equality of β's.

Let B_o and B_L be the estimators of the ordinary-weight slope β_o and the lightweight slope β_L, respectively. Then let

$Q = B_o - B_L$; It has zero mean if $\beta_o = \beta_L$.

$$\hat{\sigma}_Q^2 = \hat{\sigma}_{B_o}^2 + \hat{\sigma}_{B_L}^2 = s_{B_o}^2 + s_{B_L}^2 = (0.023)^2 + (0.031)^2 = (0.0386)^2$$

H_o: $\beta_o = \beta_L$ or $m_Q = 0$

H_1: $\beta_o \neq \beta_L$ or $m_Q \neq 0$

Accept if $|Q| \leq c$ or $|\hat{\beta}_o - \hat{\beta}_L| \leq c$

$c = \hat{\sigma}_Q t_{\alpha/2,n'} = (0.0386)t_{\alpha/2,n'}$. The proper choice of n' is not immediately evident, but it is clear that H_o will be accepted since the observed difference is only 0.162-0.173 = 0.011, or less than 1/3 of $\hat{\sigma}_Q$.

4.45 (continued)

b) One would seek paired data to permit estimating a regression of tensile strength versus compressive strength. The difficulty is in obtaining such pairs when destructive testing is involved. If within batch variation is small compared to batch-to-batch variation, these pairs can simply be specimens cast from the same batch. Each pair must then be from a different batch to obtain independent samples. If age were a systematically varying independent variable the problem would have to be treated as a multivariate one, not as simply tensile strength on compressive strength. An important test would be for $\alpha = 0$.

4.46 Let Y = April-July runoff and X = $\sum$monthly precipitations.

$\hat{\alpha} = -6.19$; $\hat{\beta} = 1.842$; $\hat{\sigma} = 1.47$

$s_A = 1.21$; $s_B = 0.151$; $\hat{\rho} = 0.883$

The analysis would be used to predict in probabilistic terms the runoff anticipated in April through July after one has observed the total precipitation in the preceding months October through April.

The correlation coefficient can be tested as on page 417, or by testing to see if the slope is different from zero. By inspection $\beta - 0 = 1.84$ is many times larger than $s_B = 0.151$. Therefore $\hat{\beta}$ and $\hat{\rho}$ are significantly different from zero.

4.47 a) Let Y = elastic limit and x = bar number

A sketch of observed values of Y versus number (diameter) and number squared ("area") suggested any non-linearity in the data is exaggerated by the squaring transformation. (Also $|r_{x,y}| > |r_{x^2,y}|$.)

$\hat{\alpha} = 66{,}350$; $\hat{\beta} = -2020$; $\hat{\sigma} = 5788$

$s_A = 4380$; $s_B = 654$; $r_{x,y} = -0.681$

Test: $H_0: \beta = 0$; $H_1: \beta \neq 0$

Accept if $|B| \leq c$

4.47 (continued)

$c = s_B t_{\alpha/2,n-2} = (654)(1.8) = 1190$ for $\alpha = 0.10$

$|\hat{\beta}| > c$, therefore reject H_o. Conclude elastic limit <u>is</u> a function of bar size.

b) Ultimate strength versus bar number.

$\hat{\alpha} = 88{,}280$; $\hat{\beta} = -1370$; $\hat{\sigma} = 5880$

$s_A = 4450$; $s_B = 664$; $r_{x,y} = -0.529$

$|\hat{\beta}| > c$, but only marginally. Conclude, tentatively, that ultimate strength is a function of bar size.

CHAPTER 5

5.1 a. The four actions are:

a_1 - do nothing

a_2 - close one month

a_3 - close three months

a_4 - reduce fee

With a_1, the states are:

θ_1 - receive job in three months with an interim cost of \$9000 and a gain equal to the expected value of the fee.

θ_2 - no job with a loss of \$9000.

With a_2, at least 8 states must be considered involving hiring problems (yes or no), receipt of a small job (yes or no), and final receipt of the job (yes or no).
Values are \$ - 6000 plus influence of jobs received.

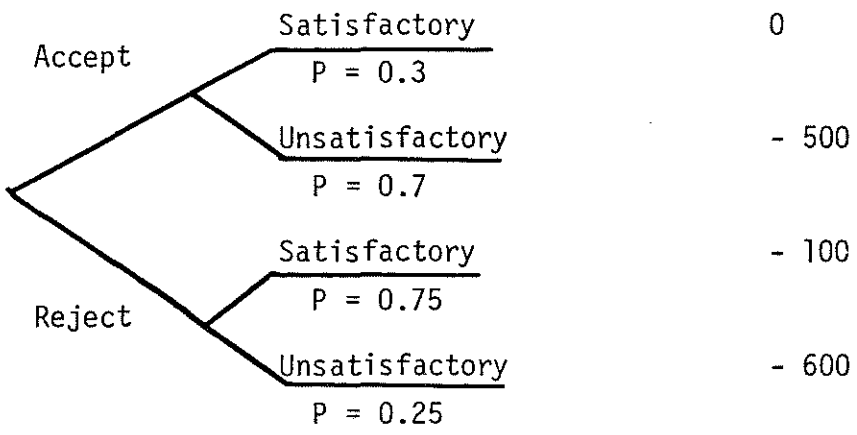
With a_3, three obvious states involve staff hiring problems (yes and no) and receipt of the job.

With a_4, the states are receipt (yes or no) of the final job and other fees.
The values are \$ - 9000 plus receipt of jobs at a reduced fee.

Note that a variety of other states are possible but only those having different values need be considered. That is, the future states could be treated as good, fair, and poor and the actual states lumped under these general classes.

5.4 With this loss structure, action a_1 (\$25/hr.) is optimum for any probability values. Action a_1 is said to "dominate" a_2 in such a case.

5.5 The decision tree is:



Action	State	Value

The design should be rejected.

5.6 If instrument indicates shallow depth, z_o is observed.

$P''[\theta_o] \propto P[z_o|\theta_o]\, P'[\theta_o] = (0.6)(0.7) = 0.42$

$P''[\theta_1] \propto P[z_o|\theta_1]\, P'[\theta_1] = (0.1)(0.3) = 0.03$

or $P''[\theta_o] = (0.42)/(0.42 + 0.03) = 14/15$

$P''[\theta_1] = 1/15$

$E[u(a_o)] = (-400)(1/15) = -26$

$E[u(a_1)] = (-100)(14/15) = -93$

The short pile (a_o) is preferred.

For a_1 to be preferred no matter what outcome is observed, we must have $E[u(a_1)] > E[u(a_2)]$ or $(-100)P''[\theta_o] > (-400)P''[\theta_1]$. For the outcome "most favoring" θ_o, i.e. for outcome z_o

$P[z_o|\theta_o]\, P[\theta_o] < 4P[z_o|\theta_1]P[\theta_1]$

$P[z_o|\theta_1] > \frac{1}{4}P[z_o|\theta_o]\, \frac{P[\theta_o]}{P[\theta_1]} = \frac{1}{4}\,(0.6)\left(\frac{0.7}{0.3}\right) = 0.35$

Note how relative consequences, relative prior probabilities, and $P[z_o|\theta_o]$ influence this result. If the test is this "unreliable" (if $P[z_o|\theta_1] > 0.35$) there is no point in making the test even if it is virtually free; its outcome can't influence the decision. (The assumption is that the ambiguous indication z_2 would remain as likely given θ_1 as it is now as the test becomes more unreliable.

5.7 If $P[\Theta = \theta_1] = p$, $P[\Theta = \theta_2] = 1 - p$,

$E[u(a_1)] = u_{11}p + u_{21}(1-p)$

$E[u(a_2)] = u_{12}p + u_{22}(1-p)$.

a_1 is preferred to a_2 if:

$E[u(a_1)] > E[u(a_2)]$ which implies

$u_{11}p + u_{21}(1-p) > u_{12}p + u_{22}(1-p)$

$u_{11}p - u_{12}p > u_{22}(1-p) - u_{21}(1-p)$

$(u_{11} - u_{12})p > (u_{22} - u_{21})(1-p)$

$\ell_{12}p > \ell_{21}(1-p)$

$E[\ell(a_2)] > E[\ell(a_1)]$

That is, a_1 is preferred if its expected loss is smaller. Reversing the steps proves that if $E[\ell(a_1)] < E[\ell(a_2)]$, a_1 is preferred.

5.8

Prior

State	$P'[\Theta = \theta]$
θ_1(OK)	0.85
θ_2(flaw)	0.15
Sum =	1.00

Sample Likelihood, $P[z|\Theta = \theta]$

	State	
Sample	θ_1	θ_2
z_1(indicates OK)	0.9	0.2
z_2(flaw)	0.1	0.8
	1.0	1.0

Joint Measures, $P[z|\Theta = \theta]P'[\Theta = \theta]$

Sample	State θ_1	θ_2	P[z]
z_1	(0.9)(0.85)=0.765	(0.2)(0.15)=0.030	0.795
z_2	(0.1)(0.85)=0.085	(0.8)(0.15)=0.120	0.205
$P'[\Theta = \theta]$	0.850	0.150	1.000

Posterior Probabilities, $P''[\theta = \theta | z]$

Sample	State θ_1	θ_2
z_1	$\frac{0.765}{0.795} = 0.962$	$\frac{0.030}{0.795} = 0.038$
z_2	$\frac{0.085}{0.205} = 0.415$	$\frac{0.120}{0.205} = 0.585$

Assuming all indicated flaws are rewelded, the expected cost per weld of new system = $0.50 + 5.00\ P[z_2] + 30.00\ P[z_1,\theta_2] = 0.50 + 5.00(0.205) + 30.00(0.03)$ = \$2.42.

The old system cost per weld = $1.00 + 5.00\ P[\theta_2] = 1.00 + 5.00(0.15) = \$1.75$. Retain the old system.

5.9 With V assumed equal to 2.5 ft./sec. with certainty, the most likely value of z is a function of θ only ($2 + 1.2 \times 10^{-3}\theta$). An observation of z = 13 could come from θ equal to 8000, 9000, or 10,000 (with z = 13 representing an implied deviation from the best fit line of -1.4, -0.2, and +1.0 respectively.) The sample likelihoods are therefore

$P[z = 13|\theta = 8000] = 0.25$

$P[z = 13|\theta = 9000] = 0.50$

$P[z = 13|\theta = 10{,}000] = 0.25$.

Multiplying by the marginal prior on Θ and normalizing gives

Flow	$P''(\theta)$ (Ignoring V undertainty)	$P''(\theta)$ (True)
8000	0.125	0.14
9000	0.500	0
10000	0.375	0.14
11000	0	0.57
12000	0	0.14

The uncertainty in velocity V opens the possibility that z = 13 was the "result" of a flow value θ from a wider range than otherwise appeared possible (e.g. θ might have been as high as 12,000, if the velocity was higher than 2.5). Generally, different posterior probabilities of θ and hence possibly different decisions will result if such embedded variables are not recognized and treated properly.

If V as well as z is measured, more information is available on θ. In this discrete case the possible values of θ are reduced in number. For example, if in addition to observing z = 13, one also learns that V = 2,

i.e. $P''[\theta = 8000] = 1$; the optimum action is simply $F = 0.1$, $\theta = 800$.

5.10 Within the discrete approximation, without scatter, if $z = 13$, θ must be 11,000. If $z = 14$, θ can be either 8000 or 12,000(depending on whether $V = 2$ or 3) with equal posterior probabilities if scatter is not considered. With scatter and $z = 14$, θ can be 8000, 9000, 11,000 or 12,000 with each posterior probability equal to 0.25. Clearly the influence of dispersion in the equation is to spread the posterior distribution on θ.

5.11 The probabilities $P[z_i|e_3]$ are identical to $P[z_i|e_1]$ (page 578). Preposterior analysis of the z_0 and z_1 branches of this alternative experiment is identical to that discussed on page 577. If z_2 is observed, the first-stage posterior probabilities of state are (page 577) $P''(\theta_0) = 0.77$ and $P''[\theta_1] = 0.23$. With <u>these</u> probabilities of state, after the sonic test and an observation of z_2, the probability of finding z_0 in the second stage <u>when</u> the hole is drilled is (compare e_2 analysis, page 575)

$$P[z_0|\theta_0]P''[\theta_0] + P[z_0|\theta_1]P''[\theta_1] = (1)(0.77) + (0)(0.23) = 0.77 .$$

In a general case this result would be combined with preposterior analysis of the second stage of the experiment. In this case, that analysis is trivial (page 576) leading here to utilities of -70 no matter which outcome (z_0 or z_1) is observed upon drilling. (The outcome of the drilling obviously determines which action is taken.) The expected utility of e_3 is

$$\begin{aligned} E[u(e_3)] &= \sum P[z_i|e_3]u(e_3,z_i) \\ &= (0.45)(-48) + (0.28)(-45) + (0.27)(-70) \\ &= -53. \end{aligned}$$

This two-stage experiment is better than a simple sonic test (e_1 with expected utility -60) but not so good as e_2, simple drilling immediately ($E[u(e_2)] = -50$). If an additional time cost of -50 units is added to this two-stage experiment, it becomes the least attractive of the four possibilities.

5.12 The discussion must specifically identify the difference between value to the engineer (responsibility) and value to the client (money).

5.13 With alternate a, the weighed reliability is (0.98)(0.6) + (0.99)(0.4) = 0.984 while that with alternate b is (0.99)(0.6) + (0.98)(0.4) = 0.986. Thus the second design, b, should be used.

5.14 a. With the pile foundation,

Settlement, δ	Probability	Utility, $
$\delta \leq 1$ in.	0.5	(0.5)(0) + (0.5)(3000) = 1500
1 in. $< \delta \leq 2$ in.	0.3	" " " " = 1500
2 in. $< \delta < 3$ in.	0.19	" " " " = 1500
$\delta = 3$ in.	0.01	- 1250

If the settlement is 3 in. with the pile foundation, the expected cost of legal problems is (0.5)(1000) + (0.5)(10,000) = $5500, and is subtracted from the fee of $3000 assuming the job is received. If the job is not received, no legal difficulties are involved. Thus the utility of finding a settlement of 3 in. is (0.5)(0) + (0.5)[3000 - 5500] = -1250. The utility of the pile foundation is (0.5)(1500)+(0.3)(1500)+(0.19)(1500)+(0.01)(-1250) = 1462.5. With the raft foundation,

Settlement, δ	Probability	Utility
$\delta \leq 2$ in.	0.50	3000
2 in. $< \delta < 3$ in.	0.41	3000
$\delta \geq 3$ in.	0.09	3000 - 5500 = -2500

The expected utility of the raft foundation is

(0.5)(3000) + (0.41)(3000) + (0.09)(-2500) = $2505.

The raft foundation should be used.

b. See if future jobs with this client are possible. If no future jobs are considered and the fee adjustment will not influence future jobs, sue if the offered fee is less than $2500.

5.15 The two PDF on cost have the following properties:

	Conventional	Push-up
Mean, Mode	x_o	$0.8x_o$
Lower limit	$0.83x_o$	$0.46x_o$
Upper limit	$1.17x_o$	$1.14x_o$

a. Utility function is zero up to $1.1x_o$ and is negative and linear from that point on. The expected loss (found by integration) is greater with the conventional design and thus the push-up design is optimum.

b. The utility function varies linearly from a value of $0.07x_o$ at 0.4 of conventional cost to $0.02x_o$ at the cost of conventional construction, x_o, and is constant at $0.02x_o$ for costs in excess of x_o up to $c = 1.10x_o$. It decreases linearly from $0.02x_o$ at 45 degrees for cost in excess of $c = 1.10x_o$. The expected utility of underwriting is positive.

5.16 The payoff table is:

θ	$P'(\theta)$	Actions	
		a_1	a_2
θ_1	0.4	0	-10
θ_2	0.2	-2	0
θ_3	0.4	-5	0

i. $u(a_1) = -2.4$
$u(a_2) = -4$
Action a_1 is optimum.

ii. The sampling reliability information is:

Sample	$P(\text{Sample}\mid\theta)$	
	θ_1	θ_2
z_1(indicates θ_1)	0.9	0.3
z_2(indicates θ_2)	0.1	0.7

The consultant found z_1 so that the posterior probabilities are found by:

State	Prior	Posterior
θ_1	0.67	$\frac{(0.9)(0.67)}{(0.9)(0.67) + (0.3)(0.33)} = 0.86$
θ_2	0.33	$\frac{(0.3)(0.33)}{(0.9)(0.67) + (0.3)(0.33)} = 0.14$

action a_1 remains optimum.

5.18 a. Integrate the product of the utility function and the density function, $f(\theta) = \frac{1}{20} \exp\left(-\frac{\theta}{20}\right)$, from 24 to infinity. The expected value is -602.

5.19 $E[u(\theta)] = \int_a^b u(\theta) f(\theta) d\theta$

$$= \int_a^b [u_b - (u_b - u_a)(\frac{b-\theta}{b-a})^m \frac{1}{B} \frac{1}{(b-a)^{t-1}} (\theta - a)^{r-1} (b - \theta)^{t-r-1} d\theta$$

Combining exponents

$$E[u(\theta)] = u_b - (u_b - u_a)\frac{B'}{B}$$

$$B = \frac{\Gamma(r)\Gamma(t-r)}{\Gamma(t)} \quad , \quad B' = \frac{\Gamma(r)\Gamma(m+t-r)}{\Gamma(m+t)}$$

Using Fig. 5.1.9, a reasonable beta distribution on θ has parameters, $r = 5$, $t = 8$. For $u_a = -10$, $u_b = 1$, and $m = 3$,

$$E[u(\theta)] = 1 - 11(\frac{1}{12}) = 0.083$$

5.20 For any design with initial capacity r the cost is the initial cost I_r plus that of the added capacity (if any) necessary to meet the uncertain future demand, X. The estimated demand is 10 million gal/day. The demand has a mean of (0.88)(10) = 8.8 million gal/day with a standard deviation of (0.12)(10) = 1.2 million gal/day. The expected added cost (if capacity is provided to just meet demand X) is

$$E[C_{add}] = \int_r^\infty 150{,}000\ (x - r)\ f_X(x)\ dx.$$

For each r, this relationship can be solved approximately by replacing $f_X(x)$ by an approximate PMF on discrete values, say 7, 8,..., 12 million gal/day. This might be an exact solution if capacity is only added in this fashion. More elegantly,

$$\int_r^\infty (x - r)\ f_X(x)dx = \int_r^\infty x f_X(x)dx - r\int_r^\infty f_X(x)dx = \int_r^\infty x f_X(x)dx - r[1 - F_X(r)].$$

The first term is (see Problem 6.22)

$$\int_r^\infty x \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2\sigma^2}(x - m)^2\right] dx = \sigma^2 f_X(r) + m_X[1 - F_X(r)].$$ Therefore

$$E[C_{add}] = \$150{,}000\{\sigma_X^2 f_X(r) - (r - m_X)[1 - F_X(r)]\}.$$

For example, for r = 7 million gal/day,

$$E[C_{add}] = (150{,}000)\left\{(1.2)^2 f_X(7) - (7 - 8.8)[1 - F_X(7)]\right\}$$

$$E[C_{add}] = (150{,}000)\left\{(1.2) f_U[(7 - 8.8)/1.2] - (7 - 8.8)[1 - F_U((7-8.8)/1.\right.$$

Solving for the six values of r, the results are:

r	I_r	$E[C_{add}]$	Total
7	$777,000	$277,000	$1,054,000
8	864,000	147,000	1,011,000
9	945,000	58,000	1,003,000
10	1,020,000	15,000	1,035,000
11	1,100,000	2,000	1,102,000
12	1,175,000	0	1,175,000

5.21 The decision tree has three action branches: report unsafe; report adequate for office use, and report adequate for storage. If the unsafe report is given, the loss is $5000 regardless of the actual condition. The chance of the structure being adequate for office use is 0.8 and the value of such an action-state combination is (5000)(0.2) + (0)(0.8) = $1000. The chance of its being inadequate for office use is 0.2 with an associated action state value of (5000)(0.2) - 5000 = $ - 4000. The expected value of a report of safe for office use is (0.8)(1000) + (0.2)(-4000) or zero. The report of adequate for storage use is associated with a gain of $5000 if this proves correct with a probability of 0.1 and a loss of +5000 - 10,000 = $ - 5000 if the structure proves inadequate (P = 0.9). The expected value of a report of acceptable for storage use is $ - 4000.

The optimum action is to report adequate or inadequate for office occupancy depending on the utility of a possible gain of $1000 versus a possible net loss of $4000. The $1000 fee does not enter the problem.

5.23 a. The ordinates of the probability density function for means of 3000 and 3500 are:

	m = 3000		m = 3500	
x	$u = \frac{x-m}{\sigma}$	$f_U(u) = \sigma f_X(x)$	$u = \frac{x-m}{\sigma}$	$f_U(u) = \sigma f_X(x)$
3000	0	0.3989	$-\frac{500}{450} = -1.11$	0.2155
3300	$\frac{300}{450} = +0.67$	0.3187	$-\frac{200}{450} = -0.445$	0.3613
2800	$-\frac{200}{450} = -0.445$	0.3613	$-\frac{700}{450} = -1.55$	0.1200
3500	$\frac{500}{450} = +1.11$	0.2155	0	0.3989

Since σ is the same for both cases, it can be ignored.

Likelihood (mean = 3000) = (0.3989)(0.3187)(0.3613)(0.2155) = 0.0099

Likelihood (mean = 3500) = (0.2155)(0.3613)(0.1200)(0.3989) = 0.0037

Relative Likelihood = 0.725 to 0.275, m = 3000 to m = 3500.

b. Posterior probabilities:

θ	$P'(\theta)$	Sample Likelihood	Product	$P''(\theta)$
3000	0.6	0.725	0.435	0.80
3500	0.4	0.275	$\frac{0.110}{0.545}$	0.20

The student should verify that it was not necessary to normalize the sample likelihoods to 0.725 and 0.275 to find these posterior probabilities. The likelihoods were normalized here to aid in understanding the implications in each step.

c. The expected value of a_0 is -8 and that of a_1 is -20. The optimum action is to do nothing.

d. Indifference is found at $u(a_1, \theta_0) = -40$.

5.24 The simplest approach to this problem is to note that the preposterior analysis begins with a prior and a sample likelihood given the state. With only two possible states, θ_0 and θ_1, a sample must indicate θ_0 or θ_1 and thus z_0 and z_1 indicating θ_0 and θ_1 respectively must be the sample result regardless of the sample size. For a sample of size one, one might decide that any sample result, x, equal to or less than 3250 psi indicates that the true mean is 3000 psi and conversely any x equal to or greater than 3250 psi indicates that the true mean is 3500 psi.

Prior

State	P'(θ)
$\theta_o(3500)$	0.4
$\theta_1(3000)$	0.6

$P(z|\theta)$

	θ_o	θ_1
z_o(indicates θ_o)	1 - F[3250|m = 3500]	1 - F[3250|m = 3000]
z_1(indicates θ_1)	F[3250|m = 3500]	F[3250|m = 3000]

Numerically, $P(z|\theta)$

z_o	0.71	0.29
z_1	0.29	0.71

Joint Measures, $P'(\theta)P(z|\theta)$

	θ_o	θ_1	P(z)
z_o	(0.4)(0.71)=0.284	(0.6)(0.29)=0.174	0.458
z_1	(0.4)(0.29)=0.116	(0.6)(0.71)=0.426	0.542
P'(θ) =	0.400	0.600	1.00

Posterior Probabilities: $P''(\theta|z)$

	θ_o	θ_1
z_o	$\frac{0.284}{0.458} = 0.62$	$\frac{0.174}{0.458} = 0.38$
z_1	$\frac{0.116}{0.542} = 0.214$	$\frac{0.426}{0.542} = 0.786$

If the utilities of 5.23 are employed in the decision tree and the cost of sampling accounted for at the e_1 level, the following are found:

$E[u(e_1,z_o,a_o)] = -3.8$

$E[u(e_1,z_o,a_1)] = -62$

a_o is optimum.

$E[u(e_1,z_1,a_o)] = -7.86$

$E[u(e_1,z_1,a_1)] = -21.4$

a_o is optimum.

$E[e_1,z_o] = -3.8$

$E[e_1,z_1] = -7.86$

$E[e_1] = (-3.8)(0.458) + (-7.86)(0.542) = -7.01.$

Accounting for the cost of sampling, the preposterior expected value of a sample of size one is -7.01 - 2 = - 9.01.

If a sample of size n is now considered, it is obvious that $\bar{x}$ replaces x in the previous discussion and z_0 and z_1 are defined as before in terms of $\bar{x}$ and 3250 psi. The $P[z|\theta]$ calculations are made as before except that, for example,

$$1 - F[3250|m = 3500, \sigma_m = \frac{\sigma}{\sqrt{n}}],$$

that is, the probability measures are calculated using $\frac{\sigma}{\sqrt{n}}$ rather than σ.

CHAPTER 6

6.1 a) The density function for w is gamma in form.

$$f_W(w) = N\, W^n e^{-\overline{x^{-k}} nw}$$

$$n = 10$$

$$\overline{x^{-k}} = 1.73 \times 10^{-16}$$

$$f_W(w) = N_o w^{10} e^{-(1.73\times10^{-16})(10)w}$$

Gamma density function:

$$f_X(x) = \frac{\lambda(\lambda x)^{k-1}}{\Gamma(k)} e^{-\lambda x}$$

$$m_X = \frac{k}{\lambda}$$

Comparing functions yields for the mean of W.

$$m_W = \frac{11}{1.73\times10^{-15}}$$

Use the mean value of W as the point estimate of the parameter, u^k. Then according to the exceedance probability of 0.02,

$$0.98 = e^{-(\frac{x}{u})^{-k}}$$

$$u^k = m_W$$

$$k = 9$$

Solve for x using tables or a series expansion of the exponential to find x = 88 mph.

b) The probability of finding n maximums is the product

$$F_X(x_1)F_X(x_2) \ldots F_X(x_\beta)$$

In which, $F_X(x) = \exp(-x^{-k}u^k)$

6.1 (continued)

The probability of Y is then:

$$F_Y = F_X^{\beta}(y) = \{\exp(-y^{-k}u^k)\}^{\beta}$$

$$= \exp(-\beta y^{-k}u^k)$$

Then using the density function of W from a.

$$\tilde{F}_Y(y) = \int F_Y f(u^k)du^k$$

$$= \int \exp(-\beta y^k u^k) N_o(u^k)^n e^{-\overline{x^{-k}}nw} du^k$$

Solving,

$$\tilde{F}_Y(y) = \left(\frac{n\overline{x^{-k}}}{\beta y^{-k} + n\overline{x^{-k}}}\right)^{n+1}$$

c) y_o = 119 mph

6.2 a) $E[\text{Cost}] = dn + c\int(\bar{x}-m)^2 f(m)dm$

$$= dn + c\sigma_M^2$$

$$= dn + c\frac{\sigma^2}{n}$$

Minimum at $dn = c\frac{\sigma^2}{n}$

$$n = \sqrt{\frac{c}{d}}\,\sigma$$

b) Equation on page 618,

$$\sigma_M''^2 = \frac{1}{\frac{1}{\sigma_M'^2} + \frac{1}{\frac{\sigma_X^2}{n}}}$$

Solve as in (a), using

$$dn = c\,\sigma_M''^2$$

c) See footnote page 579

6.2 (continued)

d)

$$E[\text{Cost}] = 20d + c_1 \frac{\sigma_1^2}{n_1} + c_2 \frac{\sigma_2^2}{n_2}$$

$$c_1 = 2c_2$$

$$\sigma_1 = 4000$$

$$\sigma_2 = 2000$$

$$n_1 + n_2 = 20$$

Optimum at $n_1 = 15$

6.3 The third interval has a probability of 0.8 of one hurricane event and of 0.2 of no event. The first interval has two separate independent descriptions each with probability of 0.1 of occurence and 0.9 of non-occurence.

Add four levels of occurrence, 0, 1, 2, 3. Three occurrences are found with probability (0.8)(0.1)(0.1) = 0.008. Two events can be found by (1, 1, 0), (1, 0, 1), and (0, 1, 1) with separate probabilities of 0.072, 0.072, and 0.002 respectively making P[2 events] = 0.146. Similarly one event can be found as (1, 0, 0), (0, 1, 0), and (0, 0, 1) with probabilities of 0.648, 0.018, and 0.018 totaling to 0.684. No hurricanes is found with (0, 0, 0), P = 0.162.

The example of 6.2.1 is solved for each of the four observations and the total expected value is equal to the sum of the products of the value given the observation by the probability of the occurrence. With continuous value relationships, it is possible to use a single expected observation level and make a single solution.

6.6 a) $\bar{x}' = 4200$ psi

$$\sigma'_M = \frac{600}{\sqrt{20}} = 134 \text{ psi}$$

$P[M \geq 3500] = 1.00$

6.6 (continued)

b) $n'' = n' + n = 25$

$n''\bar{x}'' = n\bar{x} + n'\bar{x}' = 104{,}750$

$\bar{x}'' = 4190$ psi

$(n''-1)s''^2 + n''\bar{x}''^2 = [(n-1)s^2+n\bar{x}^2] + [(n'-1)s'^2+n'\bar{x}'^2]$

$s''^2 = 319{,}558$

$\sigma_M^2 = s''^2 \frac{n''-1}{n''(n''-2)} = 13{,}338$

c) Use $t = 1.318$, $m_{90} = 4190 - (1.318)(115.5)$

$= 4038$ psi

d) Ignoring the prior, $m_{90} = 3790$ psi

One-sided confidence limit, $t = 1.533$, $m = 3840$ psi

6.7 $1 - F(x) = kx^{-\beta}$ $\qquad x \geq x_o,\ \beta > 0$

$k = x_o^{\beta}$

$1 - F(x) = x_o^{\beta}x^{-\beta}$

$F(x) = 1 - x_o^{\beta}x^{-\beta}$

$f(x) = \beta x_o^{\beta}x^{-\beta-1}$

$L(x_1, x_2, \ldots x_n) = \beta^n x_o^{\beta n} x_1^{-\beta-1}x_2^{-\beta-1}\ldots x_n^{-\beta-1}$

$= \beta^n x_o^{\beta n}(x_1x_2\ldots x_n)^{-\beta-1}$

a) $L'(\beta|\text{sample}, x_o) = k'\beta^{n'}x_o^{\beta n'}(x_1' \ldots x_n')^{-\beta-1}$

b) $L(\beta|\text{sample}, x_o) = k\beta^n x_o^{\beta n}\ (x_1 \ldots x_n)^{-\beta-1}$

$L''(\beta|\text{sample}, x_o) = k''\beta^{n+n'}x_o^{\beta(n+n')}(x_1'' \ldots x_n'')^{-\beta-1}$

$n = n + n'$

$(x_1'' \ldots x_n'') = (x_1' \ldots x_n')(x_1 \ldots x_n)$

6.7 (continued)

c) $x_o = 10$

$(x_1' \ldots x_n') = 226{,}512$

$n' = 5$

Sample: $n = 3$

$(x_1 x_2 x_3) = 1716$

$$f''(\beta) = c\beta^8 10^8 [(226{,}512)(1716)]^{-\beta-1}$$

d) Evaluate,

$$E[\text{Loss}] = 1000 \int_{13}^{\infty} \int_{0}^{\infty} c\beta^8 10^8 [(226{,}512)(1716)]^{-\beta-1} \beta x_o x^{-\beta-1} d\beta dx$$

6.9 a) Two alternatives are to be considered with three future levels for each plan. First compute the present worth of operation at the three levels for 45 years accounting for linear growth. Compute expected value for each plan and add the initial construction cost to determine which alternative is optimum.

b) Repeat (a) for the added cost and compute the expected cost of operation.

6.10 Equation 6.3.3

$$\tilde{p}_X(x) = \frac{t^x t_o^{r_o+1}}{(t_o+t)^{r_o+x+1}} \frac{(x+r_o)!}{x!\Gamma(r_o+1)}$$

$t = \frac{1}{6}$ year, $x = 0$

Velocity	Probability, p(0)
40-50 mph	0.20
50-60 mph	0.44
>60 mph	0.74

6.11 Use Equation 6.3.3 as in Problem 6.10. Analysis is similar to that of Problem 6.10. Note that for T-3, it is satisfactory to estimate $\lambda = \frac{80}{5}$ and even with T-2, the error is small. The computation of expected value is simplified using λ since the expected cost is the product of the mean number of events by the loss per event.

6.14 a) $$p(0) = \frac{t_o}{t_o + t}$$

With maximum likelihood, $\lambda = 0$.

b)

Event	Probability
0	0.882,353
1	0.103,806
2	0.012,213
3	0.001,437
4	0.000,169
5	0.000,020
6	0.000,001

mean = (0)p(0) + (1)p(1) + (2)p(2) + (3)p(3)...
= 1.333,25

Cost = $133,325

6.15 a) $\lambda = \frac{2}{120}$

$F_T(t) = 1 - e^{-\lambda t}$

$F_T(2) = 1 - e^{-(\frac{1}{30})} \approx 0$

b) $$F_T(t|\Sigma t, r) = 1 - \frac{1}{(1 + \frac{t_o}{\Sigma t})^{r+1}}$$

$$F_T(2|\Sigma t, r) = 1 - \frac{1}{(1 + \frac{2}{120})^3} = 0.07$$

6.16 a) Observe $x_5 = 1$ with $n = 1$

$$p(x_5' = 1 \text{ with } n_o = 1) = \frac{1}{3}$$

maximum likelihood estimate, $p_5 = 1$. This is the certain result with next storm.

b) Observe $x_5 = 1$, $x_1 = 2$ with $n = 3$.

$$p(1, 1, 0, 0, 0) = \frac{1}{12}$$

c) The typical year shows 5 successes and 5 failures. The cloud seeding resulted in 8 successes and 2 failures. Compute p(5,5) and p(8,2) given the typical year. The ratio of the two probabilities is approximately 2 to 1. Thus the cloud seeding result is unlikely if the basic process is unchanged.

6.19 $\bar{x} = 4200$

$$\sigma_M = \frac{600}{\sqrt{3}} = 346 \text{ psi}$$

$$u = \frac{4000-4200}{346} = -0.58$$

$P[M \geq 4000 \text{ psi}] = 0.72$

$E[\text{Loss}] = 1000[P(M \leq 3500) + P(M \geq 4500)]$

$P[M \leq 3500] = 0.02$

$P[M \geq 4500] = 0.19$

$E[\text{Loss}] = \$210$

6.20 The gain if the contractor accepts is 2000 - 1000N, in which N is the number of rainy days in t = 3 days. The expected gain is 2000 - 1000E[N], where $E[N] = \lambda t = 3\lambda$, but λ, the expected number of rainy days per day, is unknown. Assuming N is Poisson, using a "data-based" approach, λ is gamma distributed (Eq. 6.3.2) with $r_o = 10 + 12 + 15 + 6 = 43$ in t_o = (30 days/month) 4 months = 120. Then the expected gain is 2000 - (1000)3E[λ]. For the gamma, the

6.20 (continued)

mean is $(r_o+1)/t_o = 44/120 = 0.367\lambda$. Therefore the expected gain is

$$2000 - 1000(3)(0.367) = 2000 - 1100 = \$900$$

It appears to be a good gamble (if the contractor's utility and dollars are linear in this range -- between -\$1000 and \$2000). The model is convenient but not particularly accurate. Note we are ignoring the fact that the model permits two or more rainy "days" to occur in t = 1 day. (The probability of this event is only about 5%.) If this is an unacceptable error a binomial model should be used. Both of these models, however, assume independence of rainy events on different days. There is undoubtedly "persistence" of rain over such time intervals. Since we are working only with the expected value of a sum, its expected value is, <u>with</u> or <u>without</u> correlation, the sum of the expected values (i.e., $E[N] = 3\lambda = 3E[N_1$, no. of rainy days in a day] = 3P[an arbitrary day is rainy]). Therefore correlation will not influence the decision if λ is known. But the Bayesian statistical analysis of the unknown λ which was used was based on independence of the variables (through the assumed likelihood function). Positive daily correlation would effectively reduce the information content in the sample (i.e., effectively reduce t, while keeping the ratio r to t constant); implying a larger spread on the distribution on λ. Since we are interested basically in the mean of λ this would probably not influence the decision.